Asset Pricing, Fall 2007

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1. Fisher 1930 - The Theory of Interest
   Introduces a two-period model where consumer balances current consumption against consumption next period. Solving the Lagrangian yields the FOC:
   $$\frac{\partial U/\partial c_0}{\partial U/\partial c_0} = 1 + r$$

2. von Neumann & Morgenstern 1944-1953 - Theory of Games and Economic Behavior

   - St. Petersberg Paradox motivates use of log utility
   - Define preference relation over lotteries
   - Note axioms of independence and continuity
   - Expected Utility Theorem says any preference relation can be represented by maximizing $E[U]$ for some utility function.

3. Friedman, Savage 1948 - the Utility Analysis of Choices Involving Risk

   - Plot wealth vs. utiilty for a risk-averse agent
   - Define *certainty equivalent*

4. Markowitz 1952 - Portfolio Selection

   - Defines Mean-Variance optimization.
   - Illustrate mean and variance calculations for a two-asset portfolio.
   - Graph efficient frontier.

5. Arrow 1952 - The Role of Securities in the Optimal Allocation of Risk-Bearing

   - Define Arrow Securities, state prices, and complete markets
   - Define Pareto-Optimal and prove First Welfare Theorem
   - Mefine the payoff matrix M so that M_{js} is the value of asset j in state s.
   - Prove market is complete if and only if M is invertible.
   - Arrow Security prices $\mathbf{p_s}$ may be calculated from asset prices $\mathbf{p}$:

   $$\mathbf{p_s} M =^{-1} \mathbf{p}$$

6. Tobin 1958 - Portfolio Selection

- MV approach is justified when returns are Normal, or utility is quadratic
- For a portfolio $\mathbf{x}$ of risky assets with returns $\mathbf{r}$ and covariance matrix $\mathbf{\Sigma}$:

$$\begin{aligned} \mu_x &= = r_f + \mathbf{x} \cdot (\mathbf{r} - r_f) \\ \sigma_{\mathbf{x}}^{\mathbf{2}} &= \mathbf{x}'\Sigma\mathbf{x} \end{aligned}$$

- Minimizing σ_R given μ gives FOC:

$$\begin{aligned} 2\mathbf{\Sigma x} &= \lambda(\mathbf{r} - r_f) \\ \Rightarrow \mathbf{x} &= \lambda \frac{1}{2} \mathbf{\Sigma}^{-\mathbf{1}}(\mathbf{r} - r_f) \end{aligned}$$

- Two efficient portfolios can differ only in λ, so all investors buy the same proportion of risky assets (two fund separation)
- Note this analysis fails if $\mathbf{\Sigma}$ is singular (i.e. there are dependent assets)

7. Sharpe 1964 - Capital Asset Prices: A Theory of Equilibrium under Conditions of Risk

   - Assumptions: MV-optimization holds, investors can lend and borrow at r_f, investors have identical information and expectations.
   - Conclusion: all efficient portfolios lie on the *Capital Market Line*
   - If g is the tangency portfolio, we calculate the slope of the CML (Sharpe Ratio) is

$$\frac{\partial E}{\partial \sigma} = \frac{r_g - r_f}{\sigma_g}$$

   - In equilibrium, each asset's return is determined by the CAPM:

$$r_i = r_f + \frac{\sigma_{ig}}{\sigma_g^2}(r_g - r_f)$$

8. Lintner 1965 - The Valuation of Risky Assets and the Selection of Risky Investments in Stock Porftolios and Capital Budgets

   - Investors choose portfolios in two steps:
     1. Maximize θ, the portfolio's Sharpe ratio

2. Choose the fraction of wealth to invest in risky assets

- Calculate FOC for maximizing θ. This generates the CAPM again.

9. Hirshleifer 1965 - Investment Decision under Uncertainty: Choice-Theoretic Approaches

   - Two-period model with two possible states in second period, complete market for Arrow securities.
   - Individuals maximize utility: $U(c_0, c_{1a}, c_{1b})$.
   - Calculating FOCs yields state prices and interest rate:

$$P_{1a} = \frac{\partial U/\partial c_{1a}}{\partial U/\partial c_0} \quad (1)$$

$$\frac{1}{1+r} = P_{1a} + P_{1b} \quad (2)$$

   - Assuming utility function is VNM: $U = U_0 + E[U_1]$, we get

$$P_{1a} = \pi_a \frac{U_1'(c_{1a})}{U_0'(c_0)}$$

10. Pratt 1964 - Risk Aversion in the Small and in the Large

   - Given current wealth x, define the *risk premium* π of $\tilde{z}$ so that

$$u(x + E(\tilde{z}) - \pi(x, \tilde{z})) = E[u(x + \tilde{z})]$$

   - Using a Taylor expansion on both sides, calculate

$$\pi(x, \tilde{z})\frac{1}{2}\sigma_z^2 a(x)$$

   where $a(x) = -u''(x)/u'(x)$ is the *absolute risk aversion.*

   - If r is constant for all x, we show $\int e^{-\int r}$ to be a linear transformation of $u(x)$. This means it is an equivalent utility function, so r encapsulates a complete description of preferences.
   - Prove the Comparative Risk Aversion Thm:

   Theorem 1. *The following conditions are equivalent, and mean that agent 1 is more risk-averse than agent 2:*

   (a) $r_1(x) \geq r_2(x)$ *for all* $x \in \mathbb{R}$

(b) $\pi_1(x, \tilde{z} \geq \pi_2(x, \tilde{z}$ *for all* $x, \tilde{z}$

(c) $u_1(u_2^{-1}(t))$ *is concave*

Actually, these conditions mean agent 1 is *at least as* risk averse as agent 2. For strictness, the inequalities in (a) and (b) must be strict on a dense subset of x in $\mathbb{R}$, and the function in condition (c) must be *strictly* concave.

- Prove that $a(x)$ is [strictly] increasing if and only if $\pi[x]$ is [strictly] increasing. Ditto for "decreasing".
- Define relative risk aversion $r(x) = -xu''(x)/u'(x)$. Previous Thm holds if we use $x\pi(x)$ instead of $\pi(x)$.
- Show that CARA implies quadratic utility, CRRA implies log or power utility.

11. Arrow 1963 - Comment

    - Given wealth w_0, we decide how much X to invest in a risky asset. Prove:

      Theorem 1. *If absolute risk aversion increases in wealth, then* X *decreases in wealth. Similarly for decreasing or constant risk aversion.*

    - Proving the same for relative risk aversion and the fraction of wealth invested is a short extension.

12. Pye 1967 - Portfolio Selection and Security Prices