Andrew Iannaccone
Econ 203A
Rosa Matzkin
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Econometrics - HW #6: 5.4.5,11,14,17,18,25

(5.4.5) (a) By Student's Theorem, $\overline{X} \sim N\left(\mu, \sigma^2/n\right)$, so

$$\begin{aligned}\Phi(z_\alpha) &= P\left(\left|\frac{\overline{X}-\mu}{\sigma/\sqrt{n}}\right| < z_{\alpha/2}\right) \\ &= P\left(-\frac{\sigma}{\sqrt{n}}z_{\alpha/2} < |\overline{X}-\mu| < \frac{\sigma}{\sqrt{n}}z_{\alpha/2}\right)\end{aligned}$$

So the 95% confidence interval has length

$$\lambda = 2\frac{z_{.025}}{\sqrt{9}}\sigma = 2\frac{1.96}{\sqrt{9}}\sigma = 1.31\sigma$$

(b) Also by Student's Theorem,

$$\frac{\overline{X}-\mu}{S/\sqrt{n}} \sim t(n-1)$$

so the corresponding confidence interval has length

$$\lambda = 2t_{\alpha/2,n-1}\frac{s}{\sqrt{n}}$$

We can calculate the expected length using

$$E(s) = \frac{\sigma}{\sqrt{n-1}}E\left(\sqrt{\frac{(n-1)S^2}{\sigma^2}}\right)$$

The term in the radical has a $\chi^2(n-1)$ distribution, so by Thm 3.3.1,

$$E\left(\sqrt{\frac{(n-1)S^2}{\sigma^2}}\right) = \frac{2^{1/2}\Gamma\left(\frac{n-1}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{1/2}{2}\right)}$$

So for $n = 9$, the 95% confidence interval has expected length

$$E(\lambda) = \frac{2t_{.025,8}}{3}\frac{\sigma}{\sqrt{9}}\frac{2^{1/2}\Gamma\left(\frac{8}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{1/2}{2}\right)} = 1.49\sigma$$

(c) The second confidence interval is larger, since the use of the random variable S instead of σ increases uncertainty.

(5.4.11) As before, $\lambda = 2z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$. Although σ is unknown, it is bounded above by $\max\sqrt{.5(1-.5)} = .5$. So for a 90% CI with length less than .02, we require

$$.02 \leq \lambda = 2z_{.05}\frac{\sigma}{\sqrt{n}} \leq 2 \cdot 1.645\frac{.5}{\sqrt{n}}$$
$$\Rightarrow n \geq \left(\frac{1.645}{.02}\right)^2 = 6765$$

(5.4.14) (a) Inverting the terms reverses the inequalities. Multiplying through by $(n-1)S^2$ yields the desired result

(b) a and b are the 2.5% and 97.5% cutoffs for a $\chi^2(8)$ distribution, so

$$\begin{aligned} b &= 17.535 \\ a &= 2.180 \end{aligned}$$

Applying our formula, the desired confidence interval is $(3.62, 29.10)$.

(c) $X_i \sim N(\mu, \sigma^2)$, so

$$\left(\frac{X_i - \mu}{\sigma}\right)^2 \sim \chi^2(1)$$

So by Corrollary 3.3.1,

$$\sum_{i=1}^{n}\left(\frac{X_i - \mu}{\sigma}\right)^2 \sim \chi^2(n)$$

To find a confidence interval for σ^2, we must find the corresponding a and b for the $\chi^2(n)$ distribution.

(5.4.17) If X_n, Y_n are independent, then

$$M_{X_n,Y_n}(t) = M_{X_n}(t_1)M_{Y_n}(t_2)$$

If additionally $X_n \xrightarrow{D} X$ and $Y_n \xrightarrow{D} Y$, then

$$\begin{aligned} & M_{X_n}(t_1)M_{Y_n}(t_2) \xrightarrow{D} M_X(t_1)M_Y(t_2) \\ \Rightarrow\quad & M_{X,Y}(t) \xrightarrow{D} M_{X,Y}(t) \\ \Rightarrow\quad & X_n + Y_n \xrightarrow{D} X + Y \end{aligned}$$

it follows that

$$\sqrt{n}[(\overline{X} - \overline{Y}) - (\mu_1 - \mu_2)] \xrightarrow{D} W - Z,$$

where

$$W \sim N(0, \frac{1}{\lambda_1}\sigma_1^2) \qquad Z \sim N(0, \frac{1}{\lambda_1}\sigma_2^2)$$

The sum of independent normal variables is normal, and the variance is additive, so the result follows:

$$\sqrt{n}\Big[(\overline{X} - \overline{Y}) - (\mu_1 - \mu_2)\Big] \xrightarrow{D} N(0, \frac{1}{\lambda_1}\sigma_1^2 + \frac{1}{\lambda_1}\sigma_2^2)$$

(5.4.18) S_i^2 is a consistent estimator of σ_i^2, so

$$\begin{array}{rrcll}
 & S_i & \xrightarrow{P} & \sigma_i & \\
\rightarrow & S_i^2 & \xrightarrow{P} & \sigma_i^2 & \text{(by Thm 4.2.4)} \\
\rightarrow & \dfrac{S_i^2/n_i}{\sigma_i^2/n_i} & \xrightarrow{P} & 1 & \text{(by Thm 4.2.3)} \\
\rightarrow & \dfrac{S_1^2/n_1 + S_2^2/n_2}{\sigma_1^2/n_1 + \sigma_2^2/n_2} & \xrightarrow{P} & 1 & \text{(by the ratio rule)} \\
\Rightarrow & \sqrt{\dfrac{S_1^2/n_1 + S_2^2/n_2}{\sigma_1^2/n_1 + \sigma_2^2/n_2}} & \xrightarrow{P} & 1 & \text{(by Thm 4.2.4)}
\end{array}$$

(5.4.25) (a) F has an F-distribution.

(b) This is true.

(c) The probability statement is

$$P\left(a < \frac{S_2^2/\sigma_2^2}{S_1^2/\sigma_1^2} < b\right)$$

Multiplying through by S_1^2/S_2^2 yields the desired confidence interval.