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Econ 203A
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Econometrics - HW #5

3.3.15)

a) Let $\mathbf{a} = \frac{1}{n}\mathbf{1}'$, so $\overline{X} = \mathbf{aX}$. By Thm 3.5.1,

$$\begin{aligned}\overline{X} &\sim N_1(\mathbf{A}\boldsymbol{\mu}, \mathbf{A}\boldsymbol{\Sigma}\mathbf{A}') \\ &= N_1(\frac{1}{n}\sum_{j=1}^{n}\mu_j, \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij})\end{aligned}$$

b) $\overline{X} \sim N_1(\mu, \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij})$

3.5.17) Let $\mathbf{A} = [1, -2, 1]$, and define $\mathbf{Y} = \mathbf{AX}$. Then, $\mathbf{Y} \sim N_1(\mathbf{A}\boldsymbol{\mu}, \mathbf{A}\boldsymbol{\Sigma}\mathbf{A}')$ with $\mathbf{A}\boldsymbol{\mu} = 0$ and

$$\mathbf{A}\boldsymbol{\Sigma}\mathbf{A}' = (1, -2, 1)\begin{pmatrix}3 & 2 & 1\\ 2 & 2 & 1\\ 1 & 1 & 3\end{pmatrix}\begin{pmatrix}1\\ -2\\ 1\end{pmatrix} = 4$$

$$\begin{aligned}\Rightarrow P((X_1 - 2X_2 + X_3)^2 > 15.36) &= P(Y > (15.36)^{1/2}) \\ &= P\left(|Y - \mu_Y| > \sigma_Y \frac{(15.36)^{1/2}}{\sigma_Y}\right) \\ &= 2\left(1 - \Phi\left(\frac{(15.36)^{1/2}}{2}\right)\right) \\ &= 2\,(1 - \Phi(1.95)) \simeq .05\end{aligned}$$

3.5.22)

a) Using Thm 3.5.1 again,

$$\begin{aligned}\hat{\boldsymbol{\beta}} &\sim N_p((\mathbf{X}'\mathbf{X})^{-\mathbf{1}}(\mathbf{X}'\mathbf{X})\boldsymbol{\beta}, (\mathbf{X}'\mathbf{X})^{-\mathbf{1}}\mathbf{X}'\sigma\mathbf{I}[(\mathbf{X}'\mathbf{X})^{-\mathbf{1}}\mathbf{X}']') \\ &= N_p(\boldsymbol{\beta}, \sigma^2(\mathbf{X}'\mathbf{X})^{-\mathbf{1}}(\mathbf{X}'\mathbf{X})(\mathbf{X}'\mathbf{X})^{-\mathbf{1}}) \\ &= N_p(\boldsymbol{\beta}, \sigma^2(\mathbf{X}'\mathbf{X})^{-\mathbf{1}})\end{aligned}$$

b) ... and with Thm 3.5.1 yet again,

$$\begin{aligned}\hat{\mathbf{Y}} &= \mathbf{X}\hat{\boldsymbol{\beta}} \sim N_n(\mathbf{X}\boldsymbol{\beta}, \mathbf{X}\sigma^2(\mathbf{X}'\mathbf{X})^{-\mathbf{1}}\mathbf{X}') \\ &= N_n(\mathbf{X}\boldsymbol{\beta}, \sigma^2\mathbf{X}(\mathbf{X}'\mathbf{X})^{-\mathbf{1}}\mathbf{X}')\end{aligned}$$

c) Define $\mathbf{T} = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-\mathbf{1}}\mathbf{X}'$, and note $\mathbf{T}' = \mathbf{T}$, $\mathbf{T^2} = \mathbf{T}$, $\mathbf{TX} = \mathbf{X}$.

$$\begin{aligned}\hat{\mathbf{e}} &= \mathbf{Y} - \hat{\mathbf{Y}} \\ &= (\mathbf{I} - \mathbf{T})\mathbf{Y} \\ &\sim N_p((\mathbf{I} - \mathbf{T})\mathbf{X}\boldsymbol{\beta}, [\mathbf{I} - \mathbf{T}]\sigma^2\mathbf{I}[\mathbf{I} - \mathbf{T}]') \\ &= N_p((\mathbf{X} - \mathbf{TX})\boldsymbol{\beta}, \sigma^{\mathbf{2}}[\mathbf{I} - \mathbf{T}]^{\mathbf{2}}) \\ &= N_p(0, \sigma^2[\mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-\mathbf{1}}\mathbf{X}'])\end{aligned}$$

d) We can write $(\hat{\mathbf{Y}}', \hat{\mathbf{e}}')'$ as

$$\begin{pmatrix}\hat{\mathbf{Y}} \\ \hat{\mathbf{e}}\end{pmatrix} = \begin{pmatrix}\mathbf{T} \\ \mathbf{I}_p - \mathbf{T}\end{pmatrix}\mathbf{Y} = \mathbf{AY},$$

so $(\hat{\mathbf{Y}}', \hat{\mathbf{e}}')$ is normally distributed with covariance matrix

$$\begin{aligned}\mathbf{A}\boldsymbol{\Sigma}_{\mathbf{Y}}\mathbf{A}' &= \begin{pmatrix}\mathbf{T} \\ \mathbf{I}_p - \mathbf{T}\end{pmatrix}\sigma^2\mathbf{I}_p(\mathbf{T}, \mathbf{I}_p - \mathbf{T}) \\ &= \begin{pmatrix}\mathbf{T^2} & \mathbf{T}(\mathbf{I}_p - \mathbf{T})' \\ (\mathbf{I}_p - \mathbf{T})\mathbf{T}' & (\mathbf{I}_p - \mathbf{T})^{\mathbf{2}}\end{pmatrix}\sigma^2 \\ &= \begin{pmatrix}\mathbf{T} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_p - \mathbf{T}\end{pmatrix}\sigma^2\end{aligned}$$

Applying Thm 3.5.2, $\hat{\mathbf{Y}}$ and $\hat{\mathbf{e}}$ are independent.

e) We set the differential of $||\mathbf{Y} - \mathbf{Xb}||$ with respect to b to 0, which yields

$$0 = 2[\mathbf{Y} - \mathbf{Xb}]'\mathbf{X} = 2\mathbf{X}'[\mathbf{Y} - \mathbf{Xb}]$$

$$\begin{aligned}&\Rightarrow \quad \mathbf{X}'\mathbf{Y} - \mathbf{X}'\mathbf{Xb} = 0 \\ &\Rightarrow \quad \mathbf{X}'\mathbf{Xb} = \mathbf{X}'\mathbf{Y} \\ &\Rightarrow \quad \mathbf{b} = (\mathbf{X}'\mathbf{X})^{-\mathbf{1}}\mathbf{X}'\mathbf{Y}\end{aligned}$$

4.2.3) For any $\varepsilon > 0$, Chebyshev's Inequality tells us

$$P(|W_n - \mu| > \varepsilon) \leq \frac{\sigma^2}{\varepsilon} = \frac{b}{n^p \varepsilon^2}$$

Since $p > 0$, this goes to zero as $n \to \infty$, so W_n converges in probability to μ.

4.3.3) The distribution function $F(\cdot)$ is one-to-one, so

$$\begin{aligned} P(Y_n \leq y) &= P(F(Y_n) \leq F(y)) \\ &= P(n[1 - F(Y_n) \geq n[1 - F(y)]) \\ &= 1 - P(Z_n \leq n[1 - F(y)]) \end{aligned}$$

And since Y_n is the nth order statistic from $F(y)$, we also know

$$P(Y_n \leq y) = F(y)^n$$

Now define $z = n[1 - F(y)]$, so these two equations become

$$\begin{aligned} P(Y_n \leq y) &= 1 - P(Z_n \leq z) \\ P(Y_n \leq y) &= \left(1 - \frac{z}{n}\right)^n \end{aligned}$$

Combining these and taking the limit, we get

$$\lim_{n \to \infty} P(Z_n \leq z) = \lim_{n \to \infty} \left[1 - \left(1 - \frac{z}{n}\right)^n\right] = 1 - e^{-z}$$

4.3.5) For contradiction, suppose there is a random variable Y such that $Y_n \to Y$, and let $F(y)$ be its cdf. Choose some $y^0 \in \mathbb{R}$ where $F(y^0) = 0$. Note that for all $n > y^0$, $F_n(y^0) = 0$, so

$$\lim_{n \to \infty} F_n(y^0) = 0 \neq F(y^0) \rightarrow\leftarrow$$

4.4.11) Given a random variable $X \sim N(\mu, \sigma^2)$, and a function $u(X)$, we may write a Taylor expansion,

$$U(X) = u(\mu) + (X - \mu)u'(\mu) + o(X - \mu)$$

As n grows to infinity, the influence of the last term drops to zero faster than the second term, so $u(X)$ approaches $u(\mu) + (X - \mu)u'(\mu)$, which is distributed like $N(u(\mu), \sigma^2[u'(\mu)]^2)$.

In this case, $u(X) = X^3$, so

$$\begin{aligned} u(\mu) &= \mu^3 \\ u'(\mu) &= 3\mu^2 \end{aligned}$$

and $\overline{X}_n \sim N(\mu, \sigma^2/n)$, so for large n, $u(\overline{X}_n) = \overline{X}_n^3$ is approximately distributed like

$$N\left(\mu^3, \frac{9\sigma^2\mu^4}{n}\right)$$

4.4.12) We'll apply the analysis from the previous problem. Here $u(X) = \sqrt{X}$, so

$$\begin{aligned} u(\mu) &= \sqrt{\mu} \\ u'(\mu) &= \frac{1}{2\sqrt{\mu}} \end{aligned}$$

and $\frac{Y}{n} \sim N\left(\mu, \frac{\mu}{n}\right)$ for large n, so $u\left(\frac{Y}{n}\right) = \sqrt{\frac{Y}{n}}$ is approximately distributed like

$$N\left(\sqrt{\mu}, \frac{\mu}{n}\left(\frac{1}{2\sqrt{\mu}}\right)^2\right) = N\left(\sqrt{\mu}, \frac{1}{4n}\right)$$

So $\sqrt{\frac{Y}{n}}$ has variance essentially independent of μ.

4.5.1) We'll prove each direction separately:
($\Longrightarrow$) Follows immediately from Thm 4.5.5
($\Longleftarrow$) Let $\mathbf{a} = \mathbf{e_i}$. Then the ith element of X_n, X_n^i, converges in distribution to

$$X_n^i \to N_1(\mu_1, \sigma_i^2)$$

That is, we know that $\mathbf{X_n}$ converges to a multivariate normal distribution. Furthermore, the marginal distributions have the desired means and variances. It remains only to show the covariances are as desired.

Let $\mathbf{a} = \mathbf{e_i} + \mathbf{e_j}$. Then we get

$$X_n^i + X_n^j \to N_1\left(\mu_i + \mu_j, \begin{pmatrix} \sigma_{ii} & \sigma_{ij} \\ \sigma_{ji} & \sigma_{jj} \end{pmatrix}\right)$$

So the covariances are the elements of $\mathbf{\Sigma}$, as desired. We conclude that

$$\mathbf{X_n} \to N_p(\mu, \mathbf{\Sigma})$$

4.5.4) Note that we can write $(\mathbf{X_n}, \mathbf{Y_n})'$ as

$$\begin{pmatrix} \mathbf{X}'_\mathbf{n} \\ \mathbf{Y}'_\mathbf{n} \end{pmatrix} = \begin{pmatrix} \mathbf{I}_p \\ \mathbf{0} \end{pmatrix} \mathbf{X}'_\mathbf{n} + \begin{pmatrix} \mathbf{0} \\ \mathbf{I}_p \end{pmatrix} \mathbf{Y}'_\mathbf{n}$$

By Slutsky's Thm,

$$\begin{pmatrix} \mathbf{I}_p \\ \mathbf{0} \end{pmatrix} \mathbf{X_n} \xrightarrow{D} \begin{pmatrix} \mathbf{I}_p \\ \mathbf{0} \end{pmatrix} \mathbf{X} \qquad\qquad \begin{pmatrix} \mathbf{0} \\ \mathbf{I}_p \end{pmatrix} \mathbf{Y_n} \xrightarrow{D} \begin{pmatrix} \mathbf{0} \\ \mathbf{I}_p \end{pmatrix} \mathbf{Y}$$

Thm 4.3.3 says we can add convergent variables, so

$$\begin{pmatrix} \mathbf{I}_p \\ \mathbf{0} \end{pmatrix} \mathbf{X}'_\mathbf{n} + \begin{pmatrix} \mathbf{0} \\ \mathbf{I}_p \end{pmatrix} \mathbf{Y}'_\mathbf{n} \xrightarrow{D} \begin{pmatrix} \mathbf{I}_p \\ \mathbf{0} \end{pmatrix} \mathbf{X}' + \begin{pmatrix} \mathbf{0} \\ \mathbf{I}_p \end{pmatrix} \mathbf{Y}'$$

That is,

$$\begin{pmatrix} \mathbf{X}'_\mathbf{n} \\ \mathbf{Y}'_\mathbf{n} \end{pmatrix} \xrightarrow{D} \begin{pmatrix} \mathbf{X}' \\ \mathbf{Y}' \end{pmatrix}$$

Or equivalently, $(\mathbf{X_n}, \mathbf{Y_n}) \xrightarrow{D} (\mathbf{X}, \mathbf{Y})$.