Andrew Iannaccone
Econ 201 - John Riley
Due Oct 23, 2007

MICROECONOMICS - HW SET #2

1. **Preferences**

(a) For contradiction, suppose there exist two distinct maximizing values:

$$x^0, x^1 \in arg\max_x \{U(x)|px \leq I\}$$

Consider any convex combination of these values,

$$x^\lambda = (1-\lambda)x^0 + \lambda x^1 \qquad \text{with } \lambda \in (0,1)$$

The budget set defined by $px \leq I$ is convex, so it contains x^λ. But $U(\cdot)$ is strictly quasi-concave, so

$$U(x^\lambda) > \min\{U(x^0), U(x^1)\}$$

Thus, x^0 and x^1 are not maxima. $\rightarrow\leftarrow$

(b) ($\Rightarrow$) Suppose $U(\cdot)$ is strictly quasi-concave. Given any $x^0, x^1 \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$, we may define

$$x^\lambda = \lambda\, x^1 + (1-\lambda)\, x^0$$
$$x_0^\lambda = \lambda\, V(x^0) + (1-\lambda)\, V(x^1)$$

By definition,

$$\begin{aligned} U(x_0^\lambda, x^\lambda) &= x_0^\lambda + V(x^\lambda) \\ &= \underline{\lambda\, V(x^0)} + \underline{(1-\lambda)\, V(x^1)} + V(x^\lambda) \end{aligned} \tag{1}$$

and because $U(\cdot)$ is strictly quasi-concave,

$$\begin{aligned} U(x_0^\lambda, x^\lambda) &> \lambda\, U(x_0^0, x^0) + (1-\lambda)\, U(x_0^1, x^1) \\ &= \lambda(V(x^1) + \underline{V(x^0)}) + (1-\lambda)(V(x^0) + \underline{V(x^1)}) \end{aligned} \tag{2}$$

Combining (1) and (2), the underlined terms cancel, leaving

$$V(x^\lambda) > \lambda\, V(x^1) + (1-\lambda)\, V(x^0),$$

so $V(\cdot)$ is strictly concave, as desired.

($\Leftarrow$) Suppose $V(\cdot)$is strictly concave. Since x is concave, $U(\cdot)$ is the sum of two concave functions, one of which is strictly concave. $U(\cdot)$ is therefore strictly concave, and therefore strictly quasi-concave.

(c) No. One counterexample is

$$U(x_1, x_2, x_3, x_4) = \sqrt{x_1 x_2} + \sqrt{x_3 x_4}$$

For $j = 1, 2, 3, 4$, we have

$$\lim_{x_j \to \infty} \frac{\partial U}{\partial x_j}(x) \to \infty,$$

but as shown in (1.c) of the previous HW set, the optimal consumption is often a corner solution, where the consumer uses only two of the resources.

2. **Indirect utility functions and expenditure functions**

(a) Consider the transformation of the utility function,

$$U^*(x) = [U(x)]^{1/3} = x_1 + 3x_2^{1/3}$$

$U^*(x)$ is quasi-linear, so we can characterize its demand function: the consumer spends all income up to some threshold to a single good, then spends all further income to the other good.

More precisely, define the marginal utility of a dollar spent on x_1 or x_2 as

$$\lambda_1(x_1, x_2) = \frac{\partial U^*}{\partial x_1}\frac{\partial x_1}{\partial I} = \frac{1}{p_1}$$
$$\lambda_2(x_1, x_2) = \frac{\partial U^*}{\partial x_2}\frac{\partial x_2}{\partial I} = \frac{1}{p_2}x_2^{-2/3}$$

Up to some threshold I, the consumer dedicates all income to x_2 (since $\lambda_2 > \lambda_1$ for small x). Above the threshold $\lambda_1 > \lambda_2$, the consumer spends all further income on x_1. If we define the threshold to be where $x_2 = \tau$, then

$$\begin{aligned} &\lambda_1(0,\tau) = \lambda_2(0,\tau) \\ \iff &\frac{1}{p_1} = \frac{1}{p_2}(\tau)^{-2/3} \\ \iff &\tau = \left(\frac{p_1}{p_2}\right)^{3/2} \end{aligned}$$

Thus, we can write the demand function for $U^*(x)$ in terms of τ,

$$x^*(p,I) = (x_1,x_2) = \begin{cases} (0, \dfrac{I}{p_2}) & \text{for small } I \\ (\dfrac{I - p_2\tau}{p_1}, \tau) & \text{for large } I \end{cases}$$

Now recall that preferences are invariant under increasing monotonic transformations. Since $U^*(x)$ is an increasing transformation of $U(x)$, their demand functions are the same. Therefore,

$$V(p,I) = U(x^*(p,I)) = \begin{cases} \left(3\left(\dfrac{I}{p_2}\right)^{1/3}\right)^3 & (\text{for } I < p_2\tau) \\ \left(\dfrac{I - p_2\tau}{p_1} + 3\tau^{1/3}\right)^3 & \end{cases}$$

Or without the τ's,

$$V(p,I) = \begin{cases} 27\dfrac{I}{p_2} & (\text{for } I < p_1\sqrt{\dfrac{p1}{p2}}) \\ \left(\dfrac{I}{p_1} + 2\sqrt{\dfrac{p_1}{p_2}}\right)^3 & \end{cases}$$

(b) Applying duality to the results in (a), we get

$$M(p,U) = \begin{cases} \dfrac{p_2}{27}U & \left(\text{for } U < 27\left(\dfrac{p_1}{p_2}\right)^{3/2}\right) \\ p_1(U^{1/3} - 2\sqrt{p_1/p_2}) & (\text{otherwise}) \end{cases}$$

so the expenditure function is linear when $U < 27(p_1/p_2)^{3/2}$.

(c) Yes. Differentiating with respect to U yields

$$\frac{\partial M}{\partial U} = \begin{cases} \dfrac{p_2}{27} & \text{(for } U < 27(p_1/p_2)^{3/2}\text{)} \\ \dfrac{p_1}{3}U^{-2/3} & \text{otherwise} \end{cases} \tag{3}$$

Both segments are continuous, and plugging in the threshold value of U shows the function is continuous there too:

$$\frac{p_1}{3}\left(27\left(\frac{p_1}{p_2}\right)^{3/2}\right)^{-2/3} = \frac{p_1}{3}\frac{p_2}{9p_1} = \frac{p_2}{27}$$

(d) Minimizing cost pz for the firm is the same problem as minimizing expenditure px for the consumer. The cost function is therefore identical to the consumer's expenditure function, replacing U with Q.

3. **Equilibrium with identical homothetic preferences**

(a) $U(x^h)$ is a CES function, and therefore homogeneous. Since homogeneous functions are homothetic, our consumers have uniform homothetic utility functions, so equilibrium prices for the aggregate consumer yield a Walrasian equilibrium. Because $U(\omega)$ is CES, we have

$$MRS_{1,2} = \frac{1}{\delta}\left(\frac{\omega_2}{\omega_1}\right)^{1/2},$$

and the representative agent is in equilibrium only if

$$MRS_{1,2} = \frac{p_1}{p_2}$$

Combining these two conditions gives the desired result.

(b) If p_i is the price of a good in period i, then the real rate of interest is given by $(1 + r_r)\,p_1 = p_2$. Therefore,

$$\begin{aligned} r_r &= \frac{p_2}{p_1} - 1 \\ &= \delta\left(\frac{\omega_1}{\omega_2}\right)^{1/2} - 1 \end{aligned}$$

(c) The utiliity of the representative consumer is a function of ω_1, ω_2, and s,

$$U(\omega_1, \omega_2, s) = \sqrt{\omega_1 - s} + \delta\sqrt{\omega_2 + s}$$

At the optimal s, $\frac{\partial U}{\partial s} = 0$, so

$$\begin{aligned} \frac{-1}{2\sqrt{\omega_1 - s}} + \frac{\delta}{2\sqrt{\omega_2 + s}} &= 0 \\ \Rightarrow \quad \sqrt{\omega_2 + s} &= \delta\sqrt{\omega_1 - s} \\ \Rightarrow \quad \omega_2 + s &= \delta^2\omega_1 - \delta^2 s \\ \Rightarrow \quad s &= \frac{\delta^2\omega_1 - \omega_2}{1 + \delta^2} \end{aligned}$$

(d) Assuming some positive amount of arbitrage occurs, p_1 and p_2 must be in equilibrium - that is, we are not at a corner solution. Since storage is costless, there is an arbitrage opportunity if $p_1 < p_2$. Similarly, if $p_1 > p_2$, then the marginally stored unit loses money. Thus, the market can be in equilibrium with non-zero storage only if $\frac{p_1}{p_2} = 1$.

(e) There is no storage when $s \leq 0$, as calculated in (c); i.e. $\omega_2 \geq \delta^2\omega_1$.

4. **Two commodity 2 period model**

(a) Consumers have identical homothetic preferences, so we can solve for equilibrium prices using a representative agent. In equilibrium, U_i/p_i is the same for all goods, so

$$\begin{aligned} p &\sim (U_1, U_2, U_3, U_4) \\ &= (\frac{2}{\omega_{11}}, \frac{4}{\omega_{12}}, \frac{1}{\omega_{21}}, \frac{2}{\omega_{22}}) \\ &= (\frac{1}{200}, \frac{4}{100}, \frac{1}{100}, \frac{2}{25}) \\ &\sim (1, 8, 2, 16) \end{aligned}$$

(b) The market interest rate is defined by the ratio of the future spot price to the future price

$$p'_s = p_f(1 + r_m)$$

so in the notation of this problem,

$$r_m = \frac{p'_s}{p_f} - 1 = \frac{1}{p_{21}} - 1 = -\frac{1}{2}$$

Note that since money in the second period buys more than money in the first period, the interest rate is negative.

(c) By definition,

$$r_r = \frac{p_{11}}{p_{21}} - 1 = -\frac{1}{2}$$

(d) Whatever the exact amount stored, the new endowments, ω'_{12} and ω'_{22} must satisfy the conditions for Walresian equilibrium. As we found in part (a), this requires

$$\frac{4}{\omega'_{12}} = \frac{2}{\omega'_{22}}$$

Since the total amount of good 2 is unchanged,

$$\omega'_{12} + \omega'_{22} = 125$$

Solving both equations yields

$$\omega' = (400, \frac{250}{3}, 100, \frac{125}{3})$$

so $s = 50/3$ and the new price vector is

$$\begin{aligned} p' &= (\frac{1}{200}, \frac{4}{250/3}, \frac{1}{100}, \frac{2}{125/3}) \\ &\sim (1, \frac{48}{5}, 2, \frac{48}{5}) \end{aligned}$$

(e) By definition, the market interest rate is

$$r_m = \frac{p'_s}{p_f} - 1 = \frac{1}{2} - 1 = -\frac{1}{2}$$

but as shown by the price vector in (d), there is no single "real rate of interest."

Good 1: Forgoing 1 unit in 1^{st} period yields .5 units in 2^{nd}

Good 2: Forgoing 1 unit in 1^{st} period yields 1 unit in 2^{nd}