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MICROECONOMICS - HW #4

## 1. South Pacific Island

(a) Each agent has identical homothetic VNM utility

$$E(U) = \sum_s \pi_s {c_s}^{1/2},$$

so we can find prices using a representative agent. Each price $p_i$ is proportional to $U_i(\boldsymbol{\omega})$, so

$$\boldsymbol{p} = (\frac{1}{2}\pi_s \omega_s^{-1/2}) \sim (\frac{1}{4\sqrt{100}}, \frac{1}{4\sqrt{100}}, \frac{1}{2\sqrt{400}}) \sim \mathbf{1}$$

(b) Each plantation's value is equal to the value of its component state claims:

$$p^A = \sum_s p_s c_s^A = \boldsymbol{p} \cdot (0, 100, 200) = 300$$
$$p^B = \sum_s p_s c_s^B = \boldsymbol{p} \cdot (100, 0, 200) = 300$$

(c) Preferences are identical and homothetic, so each agent owns states in the same ratio. Endowments are equal in value, so

$$\boldsymbol{c}^A = \boldsymbol{c}^B = \boldsymbol{\omega}/2$$

Alex and Bev achieve this by trading for half of each-other's plantations.

(d) Having fewer independent assets than states only means we are not *guaranteed* to reach WE. Nothing prohibits it from happening in any particular case. Here the third state is already allocated optimally, and is unaffected by trading. Effectively, it's not part of the market.

**2. Insurance with a non-tradable good**

(a) As shown in Section 8.3, zero aggregate risk with homogeneous expectations implies

$$\frac{p_1}{p_2} = \frac{\pi_1}{\pi_2} = 1$$

along the 45° certainty line. So every point on the line is an equilibrium, because the indifference curves are tangent. Trading at $\boldsymbol{p} = \mathbf{1}$, the endowment leads to

$$\boldsymbol{c}_A = \boldsymbol{c}_B = (50, 50)$$

(b) Alex's expected utility is

$$E(U^A) = \frac{1}{2}\left(\sqrt{\frac{1}{4}}\sqrt{c_1^A} + \sqrt{c_1^B}\right) = \frac{3}{4}\left(\frac{1}{3}\sqrt{c_1^A} + \frac{2}{3}\sqrt{c_1^B}\right)$$

Preferences are invariant under monotonic transformations, so Alex's preferences are the same as if she had constant $h = 1$ and $\pi_1 = 1/3$. Symmetrically, Bev behaves as if $\pi_2 = 1/3$.

(c - e) Alex's preferences are CES with $\beta = 1/2$, so her demand $c_i^A$ is proportional to $(\alpha_i/p_i)^\sigma$. That is, for some $\lambda$,

$$\begin{aligned}
\boldsymbol{c}^A &= \lambda\left(\frac{\alpha_i}{p_i}\right)^\sigma = \lambda\left(\left(\frac{1/3}{p_1}\right)^2, \left(\frac{2/3}{p_2}\right)^2\right) \\
&= \lambda\frac{1}{9}\left(\frac{1}{p_1^2}, \frac{4}{p_2^2}\right) \\
&= \frac{\boldsymbol{p}\cdot\boldsymbol{c}^A}{\boldsymbol{p}\cdot\frac{1}{9}\left(\frac{1}{p_1^2}, \frac{4}{p_2^2}\right)}\frac{1}{9}\left(\frac{1}{p_1^2}, \frac{4}{p_2^2}\right) \\
&= \frac{100}{p_1(p_2 + 4p_1)}(p_2^2, 4p_1^2)
\end{aligned}$$

Alex and Bev have symmetric preferences for goods 1 and 2, so to get Bev's demand we just reverse all references to the two goods:

$$\boldsymbol{c}^B = \frac{100}{p_2(4p_2 + p_1)}(4p_2^2, p_1^2)$$

Appealing again to the symmetry of the two assets $\boldsymbol{p} = (1,1)$. This gives

$$\boldsymbol{c}^A = \frac{100}{1+4}(1,4) = (20,80)$$
$$\boldsymbol{c}^B = \frac{100}{4+1}(4,1) = (80,20)$$

So $\boldsymbol{c}^A + \boldsymbol{c}^B = (100,100) = \boldsymbol{\omega}$, so $\boldsymbol{p} = (1,1)$ clears the market as desired.

**3. Choice over time**

(a) For any $h > 0$,

$$-\frac{v''(c+h)}{v'(c+h)} < -\frac{v''(c)}{v'(c)}$$

Rearranging the terms yields

$$v''(c+h) > \frac{v''(c)}{v'(c)} v'(c+h)$$

Subtracting $v''(c)$ and dividing by $h$,

$$\frac{v''(c+h) - v''(c)}{h} > \frac{v''(c)}{v'(c)} \left[\frac{v'(c+h) - v'(c)}{h}\right]$$

In the limit as $h \to 0$,

$$v'''(c) > \frac{v''(c)}{v'(c)} v''(c)$$

Since $v'(c) > 0$, we have $v'''(c) > 0$ as desired.

(b) If $h(\cdot)$ is strictly convex, then Jensen's inequality says

$$\sum_s \pi_s h(c_s) = E(h(\boldsymbol{c})) \geq h(E(\boldsymbol{c})) = h(\bar{\boldsymbol{c}})$$

For non-trivial $\boldsymbol{c}$, the inequality is strict.

(c) Savings grow at interest rate $r$, so savng $x$ yields $(1+r)x$ in period 2:

$$\begin{aligned} U(x) &= v_1(c_1) + \delta v_2(c_2) \\ &= v_1(\omega_1 - x) + \delta v_2(\omega_2 + (1+r)x) \end{aligned}$$

(d) Differentiating $U(x)$ twice yields

$$U''(x) = v_1''(\omega_1 - x) + (1+r)^2 \delta v_2''(\omega_2 + (1+r)x)$$

We already know $v''(x) \leq 0$ for all $x$, but if $v''(x) = 0$ then $v''(x + \varepsilon) > 0$ because $v'''(x) > 0$. So $v''(x)$ must be strictly negative and $U(\cdot)$ is strictly concave.

At optimal savings, $U'(\overline{x}) = 0$, or

$$-v_1'(\omega_1 - \overline{x}) + \delta(1+r)v_2'(\overline{\omega_2} + (1+r)\overline{x} = 0$$

With multiple second-period states, expected utility becomes

$$\hat{U}(x) = v_1(\omega_1 - x) + \delta \sum_s \pi_s v_2(\omega_{2s} + (1+r)x)$$

The second derivative of $\hat{U}(\cdot)$ is negative, so the new expected utility is concave (the calculation is identical to the one for $U''(\cdot)$ above).

(e) Differentiating,

$$\hat{U}'(\overline{x}) = -v_1'(\omega_1 - \overline{x}) + (1+r)\delta \sum_s \pi_s v_2'(\omega_{2s} + (1+r)\overline{x})$$

The summation term is the expected value of a convex function, so Jensen's Inequality says

$$\begin{aligned}\hat{U}'(\overline{x}) &> -v_1'(\omega_1 - \overline{x}) + (1+r)\delta v_2'(E(\omega_{2s} + (1+r)\overline{x})) \\ &= -v_1'(\omega_1 - \overline{x}) + (1+r)\delta v_2'(\overline{\omega_2} + (1+r)\overline{x})\end{aligned}$$

This is the FOC from (d), which we know to be zero, so $\hat{U}'(\overline{x}) > 0$.

(f) Adding uncertainty means utility is no longer maximized at $\overline{x}$. Now $\hat{U}'(\overline{x}) > 0$, so the optimal $x$ is greater than $\overline{x}$. That is, uncertainty makes consumers save more.

**4. State claims v. asset market trading**

(a) Utility is Cobb-Douglas, with coefficients $\alpha_i = \pi_i^h$, so demand is

$$c_i^h = \frac{I^h \pi_i^h}{p_i}$$

We can calculate the value of each consumer's endowment,

$$I^1 = \boldsymbol{p} \cdot \frac{1}{4}(z_A^1 + z_B^1) = 15(p_1 + p_2) = 120$$
$$I^2 = 3I^1 = 360$$

So the demands are

$$C = \begin{pmatrix} c_1^1 & c_2^1 \\ c_1^2 & c_2^2 \end{pmatrix} = \begin{pmatrix} \frac{3}{4}\frac{120}{p_1} & \frac{1}{4}\frac{120}{p_2} \\ \frac{1}{4}\frac{360}{p_1} & \frac{3}{4}\frac{360}{p_2} \end{pmatrix} = \begin{pmatrix} \frac{90}{p_1} & \frac{30}{p_2} \\ \frac{90}{p_1} & \frac{270}{p_2} \end{pmatrix}$$

For market clearing, we require $\boldsymbol{c}^1 + \boldsymbol{c}^2 = \mathbf{1}' \cdot C = \boldsymbol{\omega}$. That is,

$$\left(\frac{180}{p_1}, \frac{300}{p_2}\right) = (60, 60)$$

So $\boldsymbol{p} \sim (180, 300) \sim (3, 5)$, which also satisfies $p_1 + p_2 = 8$.

The value of each asset is the sum of its component state claims,

$$\boldsymbol{p} \cdot \boldsymbol{z}^1 = 240 + 30y$$
$$\boldsymbol{p} \cdot \boldsymbol{z}^2 = 240 - 30y$$

(b) Applying these prices to the previous expression for $C$,

$$C = \begin{pmatrix} \frac{90}{3} & \frac{30}{5} \\ \frac{90}{3} & \frac{270}{5} \end{pmatrix} = \begin{pmatrix} 30 & 6 \\ 30 & 54 \end{pmatrix}$$

(c) The asset portfolio $\boldsymbol{q}^1 = (1, -1/2)$ is equivalent to the state-claim portfolio $\boldsymbol{q}^1 Z$. So with $y = 0$,

$$\boldsymbol{c}^1 = \boldsymbol{q}^1 Z = (1, -1/2) \begin{pmatrix} 40 & 20 \\ 24 & 36 \end{pmatrix} = (30, 6),$$

which is Consumer 1's optimal portfolio.

(d) We can write $C = ZQ$, so when $Z$ is invertible,

$$\begin{aligned}
Q &= Z^{-1}C \\
&= \frac{1}{8\cdot 120 - 3\cdot 120y}\begin{pmatrix} 6(6-y) & -20 \\ -6(4+y) & 40 \end{pmatrix}\begin{pmatrix} 30 & 6 \\ 30 & 54 \end{pmatrix} \\
&= \frac{1}{8\cdot 10 - 3\cdot 10y}\begin{pmatrix} 3(6-y) & -10 \\ -3(4+y) & 20 \end{pmatrix}\begin{pmatrix} 5 & 1 \\ 5 & 9 \end{pmatrix} \\
&= \frac{1}{80 - 30y}\begin{pmatrix} 80-3y & -12-3y \\ -10-3y & 20 \end{pmatrix}
\end{aligned}$$

Which gives us

$$\begin{aligned}
\boldsymbol{q}^1 &= \frac{1}{16-6y}\begin{pmatrix} 16-3y \\ -8-3y \end{pmatrix} \\
\boldsymbol{q}^2 &= \frac{1}{40-15y}\begin{pmatrix} -6y \\ 60-6y \end{pmatrix}
\end{aligned}$$

(e) Consumer 1's holding of $B$ is $q^1_B = \dfrac{-8-3y}{16-6y}$, which is negative for $y \in (-8/3, 8/3)$. Similarly, his holding of $A$ is $q^1_A = \dfrac{16-3y}{16-6y}$, which is negative for $y \in (8/3, 16/3)$.

(f) With $\hat{y} = 8/3$, there is no trading because the assets are identical:

$$Z_0 = \begin{pmatrix} 40 & 20 \\ 40 & 20 \end{pmatrix}$$

(g) As $y \to \hat{y}$, trading approaches infinity. Since sufficiently large trades contracts are be prohibitively difficult to negotiate and enforce, the WE near $\hat{y}$ is infeasible for all practical purposes.