Andrew Iannaccone
Econ 203A
Rosa Matzkin
Nov 29, 2007

Econometrics HW #7 - 5.5.3, 5.5.6, 5.5.12, 5.6.8, 5.6.9, 5.6.10, 5.7.5, 5.7.6, 6.1.11, 6.2.8ac, 6.4.1

(5.5.3)

$$\begin{aligned}
\gamma(\theta) &= P_\theta\left(X_1X_2 \geq \frac{3}{4}\right) \\
&= \int_{3/4}^{1}\int_{3/4x^2}^{1} f(x_1,x_2;\theta)dx_1dx_2 \\
&= \int_{3/4}^{1}\int_{3/4x^2}^{1} \theta x_1^{\theta-1}\theta x_2^{\theta-1}dx_1dx_2 \\
&= \int_{3/4}^{1} \theta x_2^{\theta-1}\left[x_1^\theta\right]_{3/4x^2}^{1} dx_2 \\
&= \int_{3/4x^2}^{1}\left(\theta x_2^{\theta-1} - \theta\left(\frac{3}{4}\right)^\theta x_2^{-1}\right)dx_2 \\
&= 1-\left(\frac{3}{4}\right)^\theta + \theta\left(\frac{3}{4}\right)^\theta \ln\frac{3}{4}
\end{aligned}$$

(5.5.6)

$$\begin{aligned}
\alpha &= P_{H_0}(S \leq 10) \\
&= \sum_{k=0}^{10}\binom{20}{k}(.7)^k(.3)^{20-k} \\
&= .048
\end{aligned}$$

Need to include graph

(5.5.12) (a) The sum $Y = \sum_{i=1}^{8} X_i$ has distribution $Poiss(8\mu, 8\mu)$, so

$$\begin{aligned}
\alpha &= P_{H_0}(C) \\
&= 1 - P_{H_0}(Y \leq 7) \\
&= 1 - .949 = .051
\end{aligned}$$

(b)

$$\gamma(\mu) = P_{H_0}(C) = 1 - \sum_{k=0}^{7} \frac{e^{-8\mu}(8\mu)^k}{k!}$$

(c)

$$\begin{aligned}
\mu &= .75 && \Rightarrow \gamma(\mu) = 1 - .744 = .256 \\
\mu &= 1 && \Rightarrow \gamma(\mu) = 1 - .453 = .547 \\
\mu &= 1.25 && \Rightarrow \gamma(\mu) = 1 - .220 = .780
\end{aligned}$$

(5.6.8) (a) Null hypothesis is that the campaign was ineffective, so

$$\begin{aligned}
H_0 &: p = p_0 = .14 \\
H_1 &: p > p_0
\end{aligned}$$

(b) Note that $Y \sim b(n,p)$, so if the null hypothesis holds, then

$$\begin{aligned}
\mu &= np = (590)(.14) = 82.6 \\
\sigma &= \sqrt{np(1-p)} = \sqrt{(590)(.14)(.86)} = 8.43
\end{aligned}$$

Asymptotially, $Y \sim N(\mu, \sigma^2)$, so

$$\begin{aligned}
C &= \{Y : Y \geq \mu + z_{.01}\sigma\} \\
&= \{Y : Y \geq 82.6 + (2.33)(8.43)\} \\
&= \{Y : Y \geq 102.24\}
\end{aligned}$$

(c) We observed $Y = 104$, or a z-value of

$$z_\alpha = \frac{Y - \mu}{\sigma} = 2.54,$$

for which the p-value is .0055, which is smaller than .01, so the campaign was successful at the 1% significance level.

(5.6.9) By Student's Theorem $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1)$, so c is the cutoff at .025 significance for a $\chi^2(12)$ distribution $c = 23.337$.

(5.6.10) Similarly to the previous problem, c is the .05 cutoff for an F-distribution with $r_1 = 12, r_2 = 10$: $c = 2.910$

(5.7.5) We follow the technique of Example 5.7.3, but here $n_1 = n_2 = 100$, so the expression reduces to

$$\sum_{i=1}^{k} \frac{\left[X_{ij} - \frac{1}{2}(X_{i1} + X_{i2})\right]}{\frac{1}{2}(X_{i1} + X_{i2})} \sim \chi^2(k-1),$$

which holds asymptotically as $n \to \infty$. Since $n = 200$ is large, we'll take the distribution as precise. The critical value for a $\chi^2(4)$ distribution at .05 significance level is 9.49. Evaluating the left hand side of the above equation, we get 6.36, which is less than the critical value, so we cannot reject H_0 at the .05 significance level.

(5.7.6) We follow Example 5.7.4, which says that if H_0 holds, then

$$\sum_{j=1}^{b} \sum_{i=1}^{a} \frac{\left[X_{ij} - \frac{1}{n} X_{i.} X_{.j}\right]^2}{\frac{1}{n} X_{i.} X_{.j}}$$

has approximately a $\chi^2(6)$ distribution. The .05 significance level is 12.592, and evaluating the expression yields 12.94, so we reject the null hypothesis.

(6.1.11) (a)

$$\begin{aligned}
\ln L(\theta; x) &= \sum \left[\ln f(x_i; \theta)\right] \\
&= \sum \left[\ln\left(\frac{1}{\sqrt{2\pi}\sigma}\right) - \frac{(X-\theta)^2}{2\sigma^2}\right] \\
\frac{\partial \ln L}{\partial \theta} &= \sum \frac{X - \theta}{\sigma^2} \\
\Rightarrow \hat{\theta} &= \overline{X}
\end{aligned}$$

(b) $L(\theta; x)$ is strictly concave everywhere, because

$$\frac{\partial^2 \ln L}{\partial \theta^2} = \sum \frac{-1}{\sigma^2} < 0,$$

so when $\overline{X} < 0$, then $L'(\theta, x) < 0$ at all $\theta \geq 0$. Thus, the greatest value occurs at $\theta = 0$. Thus,

$$\hat{\theta} = \begin{cases} 0 & \text{if } \overline{X} < 0 \\ \overline{X} & \text{otherwise} \end{cases}$$

(6.2.8) (a)

$$\begin{aligned}
\ln f(x;\theta) &= \ln\left(\frac{1}{\sqrt{2\pi\sigma^2}}e^{\frac{1}{2}\frac{x^2}{\sigma^2}}\right) \\
&= \ln\left(\frac{1}{\sqrt{2\pi\theta}}e^{\frac{1}{2}\frac{x^2}{\theta}}\right) \\
&= -\frac{1}{2}\ln(2\pi) - \frac{1}{2}\ln\theta - \frac{1}{2}\frac{x^2}{\theta} \\
\frac{\partial \ln f}{\partial\theta} &= -\frac{1}{2\theta} + \frac{x^2}{2\theta^2} \\
\frac{\partial^2 \ln f}{\partial 2\theta^2} &= \frac{1}{2\theta^2} - \frac{x^2}{\theta^3} \\
I(\theta) &= -\int_{-\infty}^{\infty}\left(\frac{1}{2\theta^2} - \frac{x^2}{\theta^3}\right) f(x;\theta)dx \\
&= -\frac{1}{2\theta^2} + \frac{1}{\theta^3}\int_{-\infty}^{\infty} x^2 f(x;\theta)dx \\
&= -\frac{1}{2\theta^2} + \frac{1}{\theta^3}\theta \\
&= \frac{1}{2\theta^2}
\end{aligned}$$

(b) $\mathbf{X} \sim N_n(\mathbf{0}, \theta\mathbf{I}$, so if $\hat{\theta} = \frac{1}{n}\sum \mathbf{X}_i^2$, then

$$\begin{aligned}
n\frac{\hat{\theta}}{\theta} &= \mathbf{X}'\theta\mathbf{I}\mathbf{X} \sim \chi^2(n) \\
Var(\hat{\theta}) &= \frac{\theta^2}{n^2}Var\left(\frac{\theta\hat{\theta}}{n}\right) \\
&= \frac{\theta^2}{n^2}2n = \frac{2\theta^2}{n} = \frac{1}{nI(\theta)}
\end{aligned}$$

The Rao-Cramér bound is exact, so $\hat{\theta}$ is an efficient estimator.

(c) Asymptotically, $\hat{\theta} \sim N\left(\theta, \frac{2\theta^2}{n}\right)$, so

$$\sqrt{n}(\hat{\theta} - \theta) \sim N(0, 2\theta^2)$$

(6.4.1) This is a simplified case of Example 6.4.5. Here there is only one observation, so we simply have

$$\hat{\theta} = \left(\frac{x_i}{n}\right) = \left(\frac{4}{25}, \frac{11}{25}, \frac{7}{25}, \frac{3}{25}\right)$$