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Econometrics - HW #8

**(6.3.9)** (a) Note the MLE for $\hat{\theta} = \frac{Y}{n}$, so

$$\Lambda = \frac{L(\theta_0; \boldsymbol{X})}{L(\hat{\theta}; \boldsymbol{X})} = \frac{L(\theta_0; \boldsymbol{X})}{L(Y/n; \boldsymbol{X})} = \left(\frac{\theta_0}{Y/n}\right)^Y \left(\frac{1-\theta_0}{Y/n}\right)^{n-Y}$$

$Y$ is the sum of $n$ independent $b(1, \theta)$ variables, so under $H_0$,

$$Y \sim b(n, n\theta_0)$$

(b) Asymptotially, $Y \sim N(n\theta_0, n\theta_0(1-\theta_0)) = N(50, 25)$, so the test cutoff is

$$c_1 = \mu - z_{.05}\sigma = 50 - 1.96/\sqrt{25} \approx 40$$

**(6.3.15)** First, note $X_i \sim b(1, \theta)$, so $\hat{\theta} = \overline{x}$. Let $Y = \sum x_i$.

(a)

$$\Lambda = \frac{L(\frac{1}{3}; \boldsymbol{x})}{L(\overline{x}; \boldsymbol{x})} = \frac{1}{3^n} \left(\frac{1}{\overline{x}}\right)^{n\overline{x}} \left(\frac{2}{1-\overline{x}}\right)^{n(1-\overline{x}}$$

$$-2 \ln \Lambda = 2n \left[\ln 3 + \overline{x} \ln \overline{x} + (1-\overline{x}) \ln \left(\frac{1-\overline{x}}{2}\right)\right]$$

(b) From Example 6.2.1, $I(\theta) = \frac{1}{\theta\theta - 1}$, so

$$\chi^2_W = nI(\hat{\theta})(\hat{\theta} - \theta_0)^2 = \frac{n}{\overline{x}(1-\overline{x})} \left(\overline{x} - \frac{1}{3}\right)^2$$

(c)

$$\begin{aligned} l(\theta) &= Y \ln \theta + (n - Y) \ln(1 - \theta) \\ l'(\theta) &= \frac{Y}{\theta} - \frac{n - Y}{1 - \theta} = \frac{Y - n\theta}{\theta(1 - \theta)} \\ \chi_R^2 &= \frac{l'(\theta_0)^2}{nI(\theta_0)} \\ &= \frac{(\theta_0 - Y)^2}{n\theta_0(1 - \theta_0)} \\ &= \frac{9n}{2}\left(\overline{x} - \frac{1}{3}\right)^2 \end{aligned}$$

**(6.4.4)**

$$f(x; \boldsymbol{\theta}) = \frac{\partial F}{\partial x} = \begin{cases} \theta_2 \theta_1^{\theta_2} x^{-\theta_2 - 1} & \theta_1 < x \\ 0 & \text{otherwise} \end{cases}$$

The pdf is increasing in $\theta_1$, so the likelihood is as well. So $\hat{\theta}_1$ takes its maximum possible value:

$$\hat{\theta}_1 = max\{x_i\}$$

Differentiating with respect to $\theta_2$,

$$\frac{\partial l}{\partial \theta_2} = \sum_i \left(\frac{1}{\theta_2} \ln \theta_1 - \ln x_i\right) = \frac{2}{\theta_2} + n \ln \theta_1 - \sum_i \ln x_i$$

So likelihood is maximized at $\hat{\theta}_2 = \dfrac{n}{\sum_i \ln x_i - n \ln \hat{\theta}_1}$

**(6.5.3)** The likelihood for mean and variance of a normal distribution is

$$l(\mu, \sigma^2; \boldsymbol{x}) = -\frac{n}{2}\left[\ln 2\pi + \ln \sigma^2 + \frac{\sum (x_i - \mu)^2}{n\sigma^2}\right]$$

The maximizing variance is $\hat{\sigma}^2 = \dfrac{\sum (x_i - \mu)^2}{n}$, which yields a likelihood of

$$l(\mu, \hat{\sigma}^2; \boldsymbol{x}) = -\frac{n}{2}\left[\ln 2\pi + \ln \hat{\sigma}^2 + 1\right] \tag{1}$$

(a) Define a sequence $\{Z_i\}$ as the concatenation of $\{X_i\}$ and $\{Y_i\}$. Then we can write the likelihood of $X, Y \sim N(\mu, \sigma^2)$ as

$$L(\mu, \sigma^2, \mu, \sigma^2; \boldsymbol{X}, \boldsymbol{Y}) = L(\mu, \sigma^2; \boldsymbol{Z})$$

We can then use the previous expressions for the mles:

$$\begin{aligned}
\hat{\mu} &= \frac{\sum z_i}{n_z} = \frac{n\overline{x} + m\overline{y}}{n+m} \\
\hat{\sigma}^2 &= \frac{\sum (z_i - \hat{\mu})^2}{n_z} = \frac{\sum (x_i - \hat{\mu})^2 + \sum (y_i - \hat{\mu})^2}{n+m}
\end{aligned}$$

We thereby obtain the maximal likelihood (1), so likelihood ratio is

$$\begin{aligned}
\ln \Lambda &= l(\hat{\mu}, \hat{\sigma}^2; \boldsymbol{z}) - \left[ l(\hat{\mu}_x, \hat{\sigma}_x^2; \boldsymbol{x}) + l(\hat{\mu}_y, \hat{\sigma}_y^2; \boldsymbol{y}) \right] \\
&= \frac{n}{2} \left[ \ln 2\pi + \ln \hat{\sigma}_x^2 + 1 \right] + \frac{m}{2} \left[ \ln 2\pi + \ln \hat{\sigma}_y^2 + 1 \right] - \frac{n+m}{2} \left[ \ln 2\pi + \ln \hat{\sigma}^2 + 1 \right] \\
&= \frac{n}{2} \ln \hat{\sigma}_x^2 + \frac{m}{2} \ln \hat{\sigma}_y^2 - \frac{n+m}{2} \ln \hat{\sigma}^2
\end{aligned}$$

where we have defined

$$\hat{\sigma}_x^2 = \frac{\sum (x_i - \overline{x})^2}{n} \qquad \hat{\sigma}_y^2 = \frac{\sum (y_i - \overline{y})^2}{n}$$

Exponentiating and applying the previous definitions, we get the desired result. Note $\hat{\mu} = u$.

$$\begin{aligned}
\Lambda &= \frac{(\hat{\sigma}_x^2)^{n/2} (\hat{\sigma}_y^2)^{m/2}}{(\hat{\sigma}^2)^{(n+m)/2}} \\
&= \frac{\left( \frac{\sum (x_i - \overline{x})^2}{n} \right)^{n/2} \left( \frac{\sum (y_i - \overline{y})^2}{m} \right)^{m/2}}{\left( \frac{\sum (x_i - \hat{\mu})^2 + \sum (y_i - \hat{\mu})^2}{m+n} \right)^{(n+m)/2}}
\end{aligned}$$

(b) Finding the maximal likelihood variance as in (a),

$$\begin{aligned}
\hat{\sigma}^2 &= \frac{\sum (x_i - \overline{x})^2 + \sum (x_i - \overline{y})^2}{n+m} \\
&= \frac{n\hat{\sigma}_x^2 + m\hat{\sigma}_y^2}{n+m}
\end{aligned}$$

The likelihood ratio is

$$
\begin{aligned}
\ln\Lambda &= l(\overline{x}, \hat{\sigma}^2; \boldsymbol{x}) + l(\overline{y}, \hat{\sigma}^2; \boldsymbol{y}) - l(\hat{\mu}_x, \hat{\sigma}_x^2; \boldsymbol{x} + l(\hat{\mu}_y, \hat{\sigma}_y^2; \boldsymbol{y}) \\
&= -\frac{n}{2}\left[\ln 2\pi + \ln\hat{\sigma}^2 + 1\right] - \frac{m}{2}\left[\ln 2\pi + \ln\hat{\sigma}^2 + 1\right] \\
&\quad + \frac{n}{2}\left[\ln 2\pi + \ln\hat{\sigma}_x^2 + 1\right] + \frac{m}{2}\left[\ln 2\pi + \ln\hat{\sigma}_y^2 + 1\right] \\
&= -\frac{1}{2}\left[(n+m)\ln\hat{\sigma}^2 - n\ln\hat{\sigma}_x^2 - m\ln\hat{\sigma}_y^2\right] \\
&= -\frac{n+m}{2}\left[\ln\hat{\sigma}^2 - \left(\frac{n}{n+m}\right)\ln\hat{\sigma}_x^2 - \left(\frac{m}{n+m}\right)\ln\hat{\sigma}_y^2\right]
\end{aligned}
$$

The bracketed term alone can be rewritten:

$$
\begin{aligned}
\ldots &= \ln\left(\frac{n\hat{\sigma}_x^2 + m\hat{\sigma}_y^2}{n+m}\right) - \frac{n}{n+m}\ln\hat{\sigma}_x^2 - \frac{m}{m+n}\ln\hat{\sigma}_y^2 \\
&= \ln\left(\frac{n\hat{\sigma}_x^2 + m\hat{\sigma}_y^2}{(n+m)(\hat{\sigma}_x^2)^{\frac{n}{n+m}}(\hat{\sigma}_y^2)^{\frac{m}{n+m}}}\right) \\
&= \ln\left[\left(\frac{n}{n+m}\right)\left(\frac{\hat{\sigma}_x^2}{\hat{\sigma}_y^2}\right)^{\frac{m}{n+m}} + \left(\frac{m}{n+m}\right)\left(\frac{\hat{\sigma}_y^2}{\hat{\sigma}_x^2}\right)^{\frac{n}{n+m}}\right]
\end{aligned}
$$

Noting that $F = \left(\frac{m-1}{n-1}\right)\left(\frac{\hat{\sigma}_x^2}{\hat{\sigma}_y^2}\right)$, we see the likelihood ratio is a function of $F$, $n$, and $m$ alone. The test can therefore be based on $F$, as desired.

**(6.5.8)** (a) The distribution is multinomial, so

$$
\Lambda = \frac{L(\boldsymbol{p}_0; \boldsymbol{x})}{L(\hat{\boldsymbol{p}}; \boldsymbol{x})} = \frac{\Pi p_{0i}^{x_i}}{\Pi \hat{p}_i^{x_i}} = \Pi\left(\frac{p_{0i}}{\frac{x_i}{n}}\right)^{x_i} \tag{2}
$$

(b) Expanding $\ln p_{0i}$ around $\frac{x_i}{n}$,

$$
\ln(p_{0i}) = \ln\left(\frac{x_i}{n}\right) + \frac{p_{0i} - \frac{x_i}{n}}{\frac{x_i}{n}} - \frac{\left(p_{0i} - \frac{x_i}{n}\right)^2}{2(p_i')^2},
$$

for some $p_i' \in (p_{0i}, \frac{x_i}{n})$. Substituting into (2),

$$\begin{aligned}
-2\ln\Lambda &= -2\sum x_i \left[\ln(p_{0i}) - \ln\left(\frac{x_i}{n}\right)\right] \\
&= -2\sum x_i \left[\frac{p_{0i} - \frac{x_i}{n}}{\frac{x_i}{n}} - \frac{\left(p_{0i} - \frac{x_i}{n}\right)^2}{2(p_i')^2}\right] \\
&= -2\sum n\left(p_{0i} - \frac{x_i}{n}\right) + \sum \frac{x_i\left(p_{0i} - \frac{x_i}{n}\right)^2}{(p_i')^2}
\end{aligned}$$

Since $\sum p_{0i} = 1$ and $\sum x_i = n$, the first evaluates to zero and we have the desired result.

(c) Since $\frac{x_i}{n} \xrightarrow{P} p_i$, $\frac{x_i}{np_i} \xrightarrow{P} 1$, so for large n

$$\frac{x_i}{(np_i')^2} \approx \frac{1}{np_i},$$

which yields the desired result.

**(7.2.4)** We derive and expression for the pdf given $\theta$ as a function of $\theta$ and $\sum x_i$ alone, without any reference to the individual $x_i$:

$$f(\boldsymbol{x};\theta) = \Pi\theta(1-\theta)^{x_i} = \theta^n(1-\theta)^{\sum x_i}$$

By Neyman's factorization theorem, $\sum x_i$ is a sufficient statistic for $\theta$.

**(7.3.3)** We can write the transformation as

$$\begin{aligned}
\boldsymbol{Y} = u(\boldsymbol{X}) &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \boldsymbol{X} \\
\boldsymbol{X} = w(\boldsymbol{Y}) &= \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \boldsymbol{Y}
\end{aligned}$$

The distribution of $\boldsymbol{Y}$ is therefore

$$\begin{aligned}
f_Y(\boldsymbol{y}) &= f_X(w(Y))|J| \\
&= f_x(y_1 - y_2) f_x(y_2) \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} \\
&= \frac{1}{\theta} e^{-(y_1 - y_2)/\theta} \frac{1}{\theta} e^{-y_2/\theta} \\
&= \begin{cases} \frac{1}{\theta^2} e^{y_1/\theta} & (0 < y_2 < y_1 < \infty) \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

We find the conditional distribution,

$$\begin{aligned} f_{2|1}(y_2|y_1) &= \frac{f_{1,2}(y_1, y_2)}{f_1(y_1)} \\ &= \frac{\frac{1}{\theta^2}e^{-y_1/\theta}}{\int_0^{y_1} \frac{1}{\theta^2}e^{y_1/\theta}} dy_2 \\ &= \begin{cases} \frac{1}{y_1} & (0 < y_2 < y_1 < \infty \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Since $Y_2 = X_2$, it has the same mean and variance as $X$: $\theta$ and $\theta^2$. It is therefore an unbiased estimator, so $\phi(y_1)$ is also unbiased:

$$\begin{aligned} \varphi(y_1) &= E(y_2|y_1) = \int_0^{y_1} y_2 \frac{1}{y_1} dy_2 = \frac{y_1}{2} \\ Var(\varphi) &= \frac{1}{4} Var(Y_1) = \frac{1}{4} Var(X_1 + X_2) = \frac{1}{2}\theta^2 \end{aligned}$$

**(8.1.10)** Defininining $Y = \sum x_i$, we can demonstrate the likelihood ratio is based on $Y$:

$$\begin{aligned} L(\theta; \boldsymbol{x}) = \Pi \frac{\theta^{x_k} e^{-\theta}}{x_k!} &= \frac{1}{\Pi x_i!}\theta^Y e^{-n\theta} \\ \Lambda = \frac{L(\theta'; \boldsymbol{x})}{L(\theta''; \boldsymbol{x})} &= \left(\frac{\theta'}{\theta''}\right)^Y e^{-n(\theta' - \theta'')} \end{aligned}$$

The ratio is monotonic in $Y$, so so a test $\Lambda \leq k$ is equivalent to a test $Y \geq c_1$ or $Y \leq c_2$ (depending on whether or not $\theta' > \theta''$. $Y$ has a Poisson distribution with parameter $n\theta_0$, so for $n = 10$ and $c_1 = 3$,

$$\begin{aligned} \gamma(.1) &= P(Y \geq 3|\theta_0 = .1) = 1 - .92 = .08 \\ \gamma(.5) &= P(Y \geq 3|\theta_0 = .1) = 1 - .125 = .875 \end{aligned}$$

**(8.2.9)** The distribution of $X$ is bernoilli, so as shown before,

$$\Lambda = \frac{L(\theta_1; \boldsymbol{x})}{L(\theta_2; \boldsymbol{x})} = \left(\frac{\theta_1}{\theta_2}\right)^{n\overline{x}} \left(\frac{1-\theta_1}{1-\theta_2}\right)^{n(1-\overline{x}}$$

the likelihood ratio is monotone in $\overline{x}$, so any uniformly most powerful test has the form $C = \{\boldsymbol{X}|\overline{x} \geq x\}$. Asymptotially, $\overline{x} \sim N(\theta, \frac{\theta(1-\theta)}{n})$, so the test's power is

$$\gamma(\theta) = P(C; \theta) = 1 - \Phi\left(\frac{c - \theta}{\sqrt{\theta(1-\theta)/n}}\right)$$

Rearranging this expression, we get

$$\begin{aligned}
\sqrt{n}(c-\theta) &= \sqrt{\theta(1-\theta)} \\
\Rightarrow \sqrt{n}(\theta_2-\theta_1) &= \sqrt{\theta_1(1-\theta_1)}z_{\gamma(\theta_1)} - \sqrt{\theta_2(1-\theta_2)}z_{\gamma(\theta_2)}
\end{aligned}$$

Rearranging and inserting our values,

$$x = \left(\frac{\sqrt{\frac{1}{20}\frac{19}{20}}(-1.64) - \sqrt{\frac{1}{10}\frac{9}{10}}1.28}{\frac{1}{10}-\frac{1}{20}}\right)^2 \qquad \approx 220$$

**(8.2.11)** Define the statistic $Y = u(x) = \Pi x_i$. We write the pdf as

$$f(\boldsymbol{x};\theta) = \Pi\theta x_i^{\theta-1} = \theta^n(Y)^{\theta-1}$$

By Neyman's factorization theorem, $Y$ is a sufficient stastic for $\theta$. The likelihood ratio,

$$\Lambda = \frac{f(\boldsymbol{x};\theta')}{f(\boldsymbol{x};\theta'')} = \left(\frac{\theta'}{\theta''}\right)^n Y^{\theta-\theta''}$$

is a monotone function of $Y$, so the UMP of $\theta$ has critical region $C = \{\boldsymbol{X}|Y \leq_Y\}$ for some $c_Y$.

**(8.3.7)** Referring back to (1), the mles are

$$\begin{aligned}
\hat{\theta}_1 &= \overline{x} \\
\hat{\theta}_2 &= \frac{\sum(x_i-\overline{x})^2}{n}
\end{aligned}$$

The maximal likelihoods are therefore

$$\begin{aligned}
l(\hat{\theta}_1,\hat{\theta}_2;\boldsymbol{x}) &= -\frac{n}{2}\left[\ln 2\pi + \ln\hat{\theta}_2 + 1\right] \\
l(\hat{\theta}_1,\theta_2';\boldsymbol{x}) &= -\frac{n}{2}\left[\ln 2\pi + \ln\theta_2' + \frac{\sum(x_i-\overline{x})^2}{n\theta_2'}\right] \\
&= -\frac{n}{2}\left[\ln 2\pi + \ln\theta_2' + \frac{\hat{\theta}_2}{\theta_2'}\right]
\end{aligned}$$

Writing $R = \frac{\hat{\theta}_2}{\theta'_2}$, we can write the likelihood ratio as

$$\begin{aligned} lr = \ln \Lambda &= l(\hat{\theta}_1, \theta'_2; \boldsymbol{x}) - l(\hat{\theta}_1, \hat{\theta}_2; \boldsymbol{x}) \\ &= \frac{n}{2} ln \left( \frac{\hat{\theta}_2}{\theta'_2} \right) - \frac{n}{2} \left( 1 - \frac{\hat{\theta}_2}{\theta'_2} \right) \\ &= \frac{n}{2}(1 - R + \ln R) \end{aligned}$$

Differentiating with respect to $R$,

$$\frac{\partial lr}{\partial R} = \frac{n}{2} \left( \frac{1}{R} - 1 \right)$$

So the likelihood ratio is increasing when $R < 1$ and decreasing otherwise. Note also that $lr$ ranges from 0 to $\infty$ for $R$'s on each side of 1. Therefore a UMP test's critical region can be specified in terms of $R$:

$$C = \{\boldsymbol{x} | \Lambda \leq c\} = \{\boldsymbol{x} | R \leq c_1 \text{ or } R \geq c_2\},$$

for some $c_1, c_2$. Substituting $R = \frac{\hat{\theta}_2}{\theta'_2}$ again, we see this is the desired result.