1. (a) Simplify the expression for M_{t+1} by writing $\theta = \dfrac{1-\alpha}{1-\rho}$ and taking logarithms,

$$m_{t+1} = \theta \ln \beta - \rho\theta\Delta c_{t+1} + (\theta - 1) r_{t+1}$$

Now adding r_{t+1} to each side,

$$m_{t+1} + r_{t+1} = \theta(\ln \beta - \rho\Delta c_{t+1} + r_{t+1}) \tag{1}$$

Setting this aside momentarily, note that $wc_t = \ln(X_t/C_t)$, so

$$\begin{aligned} wc_{t+1} &= \ln\left(\frac{X_{t+1}}{C_{t+1}}\right) \\ &= wc_t + \ln\left(\frac{X_{t+1}}{C_{t+1}}\right) - \ln\left(\frac{X_t}{C_t}\right) \\ &= wc_t + (x_{t+1} - c_{t+1}) - (x_t - c_t) \\ &= wc_t + x_{t+1} - x_t - \Delta c_{t+1} \end{aligned}$$

When we subsitute this into the generating function for r_{t+1}, the Δc_{t+1} terms cancel,

$$r_{t+1} = \kappa_0 + wc_t + x_{t+1} - x_t - \kappa_1 wc_t$$

Further substituting the generating function for x_{t+1}, we get

$$r_{t+1} = \kappa_0 + (1-\kappa_1)wc_t + (\rho_x - 1)x_t + \varphi\sigma_t e' \tag{2}$$

Now substitute (2) and the generating function for Δc_{t+1} into Equation (1). We obtain

$$\begin{aligned} m_{t+1} + r_{t+1} = \theta[&\ln \beta - \rho(\mu + x + \sigma_t\eta_{t+1}) + \kappa_0 \\ &+ (1-\kappa_1)wc_t + (\rho_x - 1)x_t + \varphi\sigma_t e_{t+1}] \end{aligned}$$

Applying the conjectured form for wc_t and collecting terms, this becomes

$$\begin{aligned} m_{t+1} + r_{t+1} = \theta[\,&\ln \beta + \kappa_0 - \rho\mu + (1-\kappa_1)A_0 \\ &+ \rho_x - 1 - \rho + (1-\kappa_1)A_1 \\ &+ (1-\kappa_1)A_2(\sigma_t^2 - \bar{\sigma}^2) \\ &+ \sigma_t(\varphi e_{t+1} - \rho\eta_{t+1})] \end{aligned}$$

Note that the first three lines consist of non-random terms and the final line has mean zero. It follows that $E[m_{t+1} + r_{t+1}]$ is simply the first three lines. Similarly, $V[m_{t+1} + r_{t+1}]$ is the variance of the fourth line alone. Thus,

$$\begin{aligned} E[m_{t+1}+r_{t+1}] + \frac{1}{2}Var[m_{t+1} + r_{t+1}] = \\ \theta[\ln\beta + \kappa_0 - \rho\mu + (1-\kappa_1)A_0 \\ + \rho_x - 1 - \rho + (1-\kappa_1)A_1 \\ + (1-\kappa_1)A_2(\sigma_t^2 - \bar{\sigma}^2)] + \frac{1}{2}\theta\sigma_t^2(\varphi^2 + \rho^2)] \end{aligned}$$

Adding $\frac{1}{2}\theta^2\bar{\sigma}^2(\varphi^2 + \rho^2)$ to the first term and subtracting it from the third term,

$$\begin{aligned} E[m_{t+1}+r_{t+1}] + \frac{1}{2}Var[m_{t+1} + r_{t+1}] = \\ \theta[\ln\beta + \kappa_0 - \rho\mu + (1-\kappa_1)A_0 + \frac{1}{2}\theta\bar{\sigma}^2(\varphi^2 + \rho^2) \\ + \rho_x - 1 - \rho + (1-\kappa_1)A_1 \\ + [(1-\kappa_1)A_2 + \frac{1}{2}\theta(\varphi^2 + \rho^2)](\sigma_t^2 - \bar{\sigma}^2)] \end{aligned}$$

By the Euler Equation, the LHS equals zero for all x_t and σ_t. The constant and coefficients of x_t and $(\sigma_t^2 - \bar{\sigma}^2)$ must therefore all equal zero. Solving the resulting equations, we obtain

$$\begin{aligned} A_0 &= \frac{1}{1-\kappa_1}\left[\rho\mu - \ln\beta - \kappa_0 - \frac{1}{2}\theta\bar{\sigma}^2(\varphi^2 + \rho^2)\right] \\ A_1 &= \frac{1}{1-\kappa_1}\Big(1 + \rho - \rho_x\Big) \\ A_2 &= -\frac{1}{1-\kappa_1}\left(\frac{1}{2}\right)\theta(\varphi^2 + \rho^2) \end{aligned}$$