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CES Shortcuts

A CES utility function takes the general form:

$$U(\mathbf{x}) = \left(\sum_{i=1}^{n} \alpha_i x_i^{\beta}\right)^{1/\beta}$$

where $\alpha_i > 0$, $\beta \in (-\infty, 1)$. We seek the optimal value of $\mathbf{x}$, where depending on the context, optimal means either:

- Maximizing utility for a given income, or
- Minimizing cost for a given utility

In either case, we'll use $\mathbf{x}^*$ to denote the optimal value. We use duality implicitly throughout this analysis.

Define the vector $\mathbf{x}' \in \mathbb{R}^n$ by

$$x_i' = \left(\frac{\alpha_i}{p_i}\right)^{\sigma} \tag{1}$$

where $\sigma = 1/(1-\beta)$ is the elasticity of substitution. As we will prove shortly, $\mathbf{x}^* = \lambda \mathbf{x}'$ for some scalar λ. Temporarily assuming this claim to be true, we get:

$$\begin{aligned} x_i^* &= \lambda x_i' \\ \mathbf{p} \cdot \mathbf{x}^* &= \lambda(\mathbf{p} \cdot \mathbf{x}') \\ U(\mathbf{x}^*) &= U(\lambda \mathbf{x}') = \lambda U(\mathbf{x}') \end{aligned} \tag{2}$$

Note that the third equality follows because $U(\cdot)$ is homogeneous of degree 1.

Combining the equations in (2) yields

$$\frac{\mathbf{x}^*}{\mathbf{x}'} = \frac{\mathbf{p} \cdot \mathbf{x}^*}{\mathbf{p} \cdot \mathbf{x}'} = \frac{U(\mathbf{x}^*)}{U(\mathbf{x}')} \tag{3}$$

This expression (3) is extremely powerful. With very little computation, it yields all the important solutions to CES optimization problems. For example, to find the market demand, note that $I = \mathbf{p} \cdot \mathbf{x}^*$, so

$$\mathbf{x}(I) = \mathbf{x}^* = \frac{\mathbf{p} \cdot \mathbf{x}^*}{\mathbf{p} \cdot \mathbf{x}'}\mathbf{x}' = \frac{I}{\mathbf{p} \cdot \mathbf{x}'}\mathbf{x}'$$

Substituting the definition of $\mathbf{x}'$ from (1), we immediately get the desired expression. Similarly, the expenditure function is

$$M(U) = \mathbf{p} \cdot \mathbf{x}^* = \frac{U(\mathbf{x}^*)}{U(\mathbf{x}')}\mathbf{p} \cdot \mathbf{x}' = \frac{U}{U(\mathbf{x}')}\mathbf{p} \cdot \mathbf{x}'$$

Similar arithmatic allows us to quickly find formulas for indirect utility, Hicksian demand, and shadow prices.

For completeness, we now prove our previous claim that that $\mathbf{x}^*$ is a scalar multiple of $\mathbf{x}'$. Consider the problem of maximizing utility at a given income I. For all CES functions with $\beta \neq 1$, we recall from HW #1 that for some scalar λ,

$$\begin{aligned} p_i &= \lambda \alpha_i x_i^{\beta-1} \\ \Rightarrow \qquad x_i &= \left(\frac{p_i}{\lambda \alpha_i}\right)^{\frac{1}{\beta-1}} = \left(\frac{\lambda \alpha_i}{p_i}\right)^{\sigma} \end{aligned}$$

Applying the budget constraint gives

$$\begin{aligned} I = \mathbf{p} \cdot \mathbf{x} &= \sum p_i \left(\frac{\lambda \alpha_i}{p_i}\right)^{\sigma} \\ &= \lambda^{\sigma} \sum p_i \left(\frac{\alpha_i}{p_i}\right)^{\sigma} \\ &= \lambda^{\sigma} \mathbf{p} \cdot \mathbf{x}' \\ \Rightarrow \qquad \lambda^{\sigma} &= \frac{I}{\mathbf{p} \cdot \mathbf{x}'} \\ \Rightarrow \qquad x_i &= \frac{I}{\mathbf{p} \cdot \mathbf{x}'}\left(\frac{\alpha_i}{p_i}\right)^{\sigma} = \frac{I}{\mathbf{p} \cdot \mathbf{x}'}x_i' \\ \Rightarrow \qquad \mathbf{x}^* &= \frac{I}{\mathbf{p} \cdot \mathbf{x}'}\mathbf{x}', \quad \text{as desired.} \end{aligned}$$

So that's it. Equations (1) and (3) are really all you need for computation with CES utility functions. You can skip past solving Lagrangians or tangency conditions; all you need to memorize is:

$$\boxed{x_i' = \left(\frac{\alpha_i}{p_i}\right)^\sigma}$$

$$\boxed{\frac{\mathbf{x}^*}{\mathbf{x}'} = \frac{\mathbf{p}\cdot\mathbf{x}^*}{\mathbf{p}\cdot\mathbf{x}'} = \frac{U(\mathbf{x}^*)}{U(\mathbf{x}')}}$$