Rough Solutions to 239B - HW2

1. To make applying our analysis to Problem 2 easier, we add two dummy variables to the model. For Problem 1, $\phi = \varphi_d = 1$.

$$
\begin{aligned}
\Delta c_{t+1} &= \mu + \phi x_t + \varphi_d \sigma_t \eta_{t+1} \\
x_{t+1} &= \rho_x x_t + \varphi_e \sigma_t e_{t+1} \\
(\sigma_{t+1}^2 - \bar{\sigma}^2) &= \nu_1 (\sigma_t^2 - \bar{\sigma}^2) + \sigma_w w_{t+1} \\
m_{t+1} &= \theta \ln \beta - \rho\theta \Delta c_{t+1} + (\theta - 1) r_{t+1} \\
r_{t+1} &= \kappa_0 + \Delta c_{t+1} + wc_{t+1} - \kappa_1 wc_t \\
wc_{t+1} &= A_0 + A_1 x_t + A_2 (\sigma_t^2 - \bar{\sigma}^2)
\end{aligned}
$$

Next, we find the conditional expectations for everything relevant $(\Delta c_{t+1}, x_{t+1}, \sigma_{t+1}, wc_{t+1}, m_{t+1}, r_{t+1})$. And then all the relevant variances: $\sigma_{cz}, \sigma_z^2, \sigma_c^2, \sigma_{rc}, \sigma_r^2, \sigma_{rm}$, etc. This is tedious, but it's best to do it now. Next use the Euler equation:

$$E(m_{t+1} + r_{t+1}) + \frac{1}{2} Var(m_{t+1} + r_{t+1}) = 0$$

Find this and collect terms in x_t and $(\sigma_t^2 - \bar{\sigma}^2)$. The equation must be zero for all values of x_t and $(\sigma_{t+1}^2 - \bar{\sigma}^2)$, so we get three non-linear equations. Solving these gives

$$
\begin{aligned}
A_1 &= \frac{\phi(\rho - 1)}{\rho_x - \kappa_1} \\
A_2 &= \frac{\theta}{2(\kappa_1 - \nu_1)} [(\rho - 1)^2 \varphi_d^2 + \varphi_e^2 A_1^2] \\
A_0 &= \frac{1}{\kappa_1 - 1} \left[\ln \beta + \kappa_0 + (1 - \rho)\mu + \frac{\theta}{2} \sigma_w^2 A_2^2 + \frac{\theta}{2} ((\rho - 1)^2 \varphi_d^2 + \varphi_e^2 A_1^2) \bar{\sigma}^2 \right]
\end{aligned}
$$

Again, note that for Problem 1, $\phi = \varphi_d = 1$.

To do part (b), note that choosing A_0 forces all the other unknown parameter values. So we have to solve numerically for A_0, then evaluate the other terms. I haven't done this yet. Phil got this:

$$
\begin{aligned}
\kappa_0 &= 0.0198 \\
\kappa_1 &= 1.0029 \\
A_0 &= 5.8515 \\
A_1 &= 5.1624 \\
A_2 &= -175.1020
\end{aligned}
$$

For (c), use the normality of M_t to find $E_t[e^{m_{t+1}}]$. This gives the risk free rate. Use the Euler equation to get excess returns:

$$
\begin{aligned}
r_{t,f} &= -E_t(m_{t+1}) - \frac{1}{2}Var_t(m_{t+1}) \\
E_t[r_{t+1} - r_{t,f}] &= E_t(r_{t+1}) + E_t(m_{t+1}) + \frac{1}{2}Var_t(m_{t+1})
\end{aligned}
$$

We can calculate these analytically, using the expectations and variances we calculated before. To get the calibrated values, we turn these into unconditional expectations and evaluate.

(d) is best solved by induction. The inductive hypothesis is:

$$
wc_t = \sum_{j=1}^{H} \kappa_1^{-j}(\Delta c_{t+j} - r_{t+j} + \kappa_0) + \kappa_1^{-H} wc_{t+H}
$$

In the limit, the transversality condition makes the last term disappear.

(e) First evaluate $E_t[\Delta c_{t+j}]$ and $E_t[r_{t+j}]$, then do the infinite geometric series, yielding

$$
\begin{aligned}
\Delta c_t^H &\approx \frac{1}{\kappa_1 - 1}\mu + \frac{1}{\kappa_1 - \rho_x} x_t \\
r_t^H &\approx \frac{1}{\kappa_1 - 1}[\kappa_0 + \mu + A_0(1 - \kappa_1)] + \frac{1}{\kappa_1 - \rho_x} x_t (A_1 \rho_x + 1 - \kappa_1 A_1) + \\
&\quad \frac{1}{\kappa_1 - \nu_1}(\sigma_{t+1}^2 - \bar{\sigma}^2)(A_2 \nu_1 - \kappa_1 A_1)
\end{aligned}
$$

Next calculate the variances of these (the covariance is zero). The sum is the answer. Again, evaluate with the numerical values.

2. (a) The system is basically identical to the one in Problem 1. Just change A_0 to A_0^m, etc. Note that we've already got ϕ and φ_d in the equations.
(b) For the excess returns, just turn r_{t+1} into r_{t+1}^m. Calculate the expectation and reuse the expressions from (1c).
(c) This is an identical inductive proof. Just justify using the transversality condition again.