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Empirical Asset Pricing - Business 239C - HW#2

1. We begin by evaluating a whole bunch of moments, which we'll make use of later.

$$\begin{aligned}
E_t\left[\Delta c_{t+1}\right] &= \mu + x_t \\
E_t\left[x_{t+1}\right] &= \rho_x x_t \\
E_t\left[(\sigma_{t+1}^2 - \bar{\sigma}^2)\right] &= \nu_1(\sigma_t^2 - \bar{\sigma}^2) \\
E_t\left[wc_{t+1}\right] &= A_0 + A_1\rho_x x_t + A_2\nu_1(\sigma_t^2 - \bar{\sigma}^2) \\
E_t\left[r_{t+1}\right] &= \kappa_0 + \mu + (1-\kappa_1)A_0 \\
&\quad + x_t(1 + A_1\rho_x - A_1\kappa_1) \\
&\quad + (\sigma_t^2 - \bar{\sigma}^2)(A_2\nu_1 - \kappa_1 A_1) \\
E_t\left[m_{t+1}\right] &= \theta\ln\beta - \rho\theta(\mu + x_t) + (\theta - 1)E_t\left[r_{t+1}\right] \\
E_t\left[m_{t+1} + r_{t+1}\right] &= \theta\ln\beta - \rho\theta(\mu + x_t) + \theta E_t\left[r_{t+1}\right] \\
&= \theta[\ln\beta + (1-\rho\theta)\mu + \kappa_0 + (1-\kappa_1)A_0 \\
&\quad + x_t(-\rho\theta + 1 + A_1(\rho_x - \kappa_1)) \\
&\quad + (\sigma_t^2 - \bar{\sigma}^2)(A_2\nu_1 - \kappa_1 A_1)]
\end{aligned}$$

We'll use the notation σ_{cz} to indicate $Cov(\Delta c_{t+1}, wc_{t+1})$, etc.

$$\begin{aligned}
\sigma_{cd} &= 0 \\
\sigma_z^2 &= A_1^2\varphi_e^2\sigma_t^2 + A_2^2\sigma_w^2 \\
\sigma_c^2 &= \sigma_t^2 \\
\sigma_{rc} &= \sigma_c^2 \\
\sigma_r^2 &= (1 + A_1^2\varphi_e^2)\sigma_t^2 + A_2^2\sigma_w^2 \\
\sigma_{rm} &= (\theta - 1)\sigma_r^2 - \rho\theta\sigma_c^2 \\
\sigma_m^2 &= \sigma_c^2\rho\theta(\rho\theta - 2\theta + 2) + (\theta - 1)^2\sigma_r^2 \\
\sigma_{r+m}^2 &= \sigma_r^2 + 2\sigma_{rm} + \sigma_m^2 \\
&= \sigma_r^2 + 2(\theta - 1)\sigma_r^2 - 2\rho\theta\sigma_c^2 + \sigma_c^2\rho\theta(\rho\theta - 2\theta + 2) + (\theta - 1)^2\sigma_r^2 \\
&= \rho\theta^2(\rho - 2)\sigma_c^2 + \theta^2\sigma_r^2 \\
&= \theta^2((\rho - 1)^2 + \varphi_e^2A_1^2)\sigma_t^2 + \theta^2A_2^2\sigma_w^2
\end{aligned}$$

(a) Returns and SDF are log-normal, so

$$E_t\left[m_{t+1} + r_{t+1}\right] + \frac{1}{2}V_t\left[r_{t+1} + m_{t+1}\right] = 0$$

Applying the moments derived above to this equation gives an expression of the form

$$A + Bx_t + C(\sigma_t^2 - \bar{\sigma}^2) = 0$$

This must hold for all x_t and $(\sigma_t^2 - \bar{\sigma}^2)$, which implies $A = B = C = 0$, or

$$\begin{aligned}
0 &= \ln\beta + \kappa_0 + (1 - \rho)\mu + \frac{1}{2}\theta(1 - \rho)^2\bar{\sigma}^2 \\
&\quad + (1 - \kappa_1)A_0 + \frac{1}{2}\theta A_1^2\varphi_e^2\bar{\sigma}^2 + \frac{1}{2}\theta\sigma_w^2A_2^2 \\
0 &= (1 - \rho) + A(\rho - \kappa_1) \\
0 &= \frac{1}{2}\theta(1 - \rho)^2 + \frac{1}{2}\theta A_1^2\varphi_e^2 + A_2(\nu - \kappa_1)
\end{aligned}$$

Solving for A_0, A_1, A_2, we have

$$A_0 = \frac{1}{\kappa_1 - 1}\left[\ln\beta + \kappa_0 + (1-\rho)\mu \right.$$
$$\left. + \frac{\theta}{2}(1-\rho)^2\bar{\sigma}^2 + \frac{\theta}{2}\varphi_e^2\bar{\sigma}^2 A_1^2 + \frac{\theta}{2}A_2^2\sigma_w^2\right] \quad (1)$$

$$A_1 = \frac{\rho - 1}{\rho_x - \kappa_1} \quad (2)$$

$$A_2 = \frac{\theta(1-\rho)^2}{2(\kappa_1 - \nu)}\left(1 + \frac{\varphi_e^2}{(\rho_x - \kappa_1)^2}\right) \quad (3)$$

(b) By definition,

$$\kappa_1 = \frac{e^{A_0}}{e^{A_0} - 1} \quad \text{and} \quad \kappa_0 = -\ln(e^{A_0} - 1) + \frac{e^{A_0}}{e^{A_0} - 1}A_0$$

These with equations (1,2,3) comprise a solvable system of equations. Using Matlab and Excel (independently), we found the following parameter values:

$$\begin{aligned} \kappa_0 &= 0.0198 \\ \kappa_1 &= 1.0029 \\ A_0 &= 5.8515 \\ A_1 &= 5.1624 \\ A_2 &= -175.1020 \end{aligned}$$

(c) Using the Euler Equation and the log-normality of m and r,

$$R_{f,t} = \frac{1}{E_t\left[M_{t+1}\right]} = e^{-E_t[m_{t+1}] - \frac{1}{2}V_t[m_{t+1}]}$$

Taking logs gives the risk free rate,

$$r_{f,t} = -E_t\left[m_{t+1}\right] - \frac{1}{2}V_t\left[m_{t+1}\right]$$

and subtracting this from the Euler Equation gives excess returns,

$$E_t\left[r_{t+1} - r_{f,t}\right] = -\frac{1}{2}V_t\left[r_{t+1}\right] - Cov_t[r_{t+1}, m_{t+1}]$$

Evaluating numerically yields

$$\begin{aligned} E_t\left[r_{t+1} - r_{f,t}\right] &= -\frac{1}{2}\sigma_r^2 - ((\theta-1)\sigma_r^2 - \rho\theta\sigma_c^2) \\ &= (\frac{1}{2} - \theta)((1 + A_1^2\varphi_e^2)\sigma_t^2 + A_2^2\sigma_w^2) + \rho\theta\sigma_t^2 \\ &= (\frac{1}{2} - \theta)A_2^2\sigma_w^2 + [\rho\theta + (\frac{1}{2} - \theta)(1 + A_1^2\varphi_e^2)]\sigma_t^2 \\ &= 0.00001303858 + 21.203906\sigma_t^2 \end{aligned}$$

(d) We prove by induction that

$$wc_t = \sum_{j=1}^{H} \kappa_1^{-j}(\Delta c_{t+j} - r_{t+j} + \kappa_0) + \kappa_1^{-H} wc_{t+H}$$

The claim holds for $H = 1$ because

$$\begin{aligned} &\kappa_1 wc_t = \Delta c_{t+1} + wc_{t+1} + \kappa_0 - r_{t+1} \\ &\Rightarrow wc_t = \kappa^{-1}(\Delta c_{t+1} + wc_{t+1} + \kappa_0) + \kappa_1^{-1} wc_{t+H} \end{aligned}$$

If the claim holds for $N = n$, then we simply expand the final term to yield the claim for $N = n + 1$,

$$\begin{aligned} wc_t &= \sum_{j=1}^{n} \kappa_1^{-j}(\Delta c_{t+j} - r_{t+j} + \kappa_0) + \kappa_1^{-n} wc_{t+n} \\ &= \sum_{j=1}^{n} \kappa_1^{-j}(\Delta c_{t+j} - r_{t+j} + \kappa_0) \\ &\quad + \kappa_1^{-(n+1)}(\Delta c_{t+n+1} + \kappa_0 - r_{t+n+1} + wc_{t+n+1}) \\ &= \sum_{j=1}^{n+1} \kappa_1^{-j}(\Delta c_{t+j} - r_{t+j} + \kappa_0) + \kappa_1^{-(n+1)} wc_{t+n+1}) \end{aligned}$$

Now we take the limit of our claim as $H \to \infty$. By the transversality condition, $E_t\left[wc_{t+H}\right] \to E[wc_t]$, which is finite, so for large H, the last term vanishes, and the κ_0 term approaches an infinite geometric series. That is,

$$\begin{aligned} wc_t &\approx E_t\left[\sum_{j=1}^{H} \kappa_1^{-j}(\Delta c_{t+j} - r_{t+j})\right] + \frac{\kappa_0}{\kappa_1}\frac{1}{1 - \frac{1}{\kappa_1}} \\ &= \frac{\kappa_0}{\kappa_1 - 1} + \Delta c_t^H - r_t^H \end{aligned}$$

To evaluate this decomposition, we solve for Δc_t^H and r_t^H.

$$\begin{aligned}
\Delta c_t^H &= \sum_{j=1}^{H} \kappa_1^{-j} E_t\left[\Delta c_{t+j}\right] \\
&= \sum_{j=1}^{H} \kappa_1^{-j} (\mu + E_t\left[x_{t+j-1}\right]) \\
&= \sum_{j=1}^{H} \kappa_1^{-j} (\mu + \rho_x^{j-1} x_t)
\end{aligned}$$

For large H and $\kappa_1 > \rho_x$, this power series converges to

$$\Delta c_t^H \approx \frac{1}{\kappa_1 - 1}\mu + \frac{1}{\kappa_1 - \rho_x} x_t \tag{4}$$

A similar power series summation yields

$$r_t^H \approx \frac{\kappa_0 + \mu}{\kappa_1 - 1} - A_0 + x_t \left(\frac{1}{\kappa_1 - \rho_x} - A_1 \right) - (\sigma_t^2 - \bar{\sigma}^2) A_2 \tag{5}$$

Evaluating at the given parameters yields

$$\begin{aligned}
\Delta c_t^H &\approx 1.560387 + 15.487092 x_t \\
r_t^H &\approx 2.5593 + 10.324692 x_t + 175.1020(\sigma_t^2 - \bar{\sigma}^2)
\end{aligned}$$

(e) Using equations (4,5), we compute the unconditional variances,

$$\begin{aligned}
Var[\Delta c_t^H] &= \left(\frac{1}{\kappa_1 - \rho_x} \right)^2 Var[x_t] = \left(\frac{1}{\kappa_1 - \rho_x} \right)^2 \frac{\varphi_e^2 \bar{\sigma}^2}{1 - \rho_x^2} \\
Var[r_t^H] &= \left(\frac{1}{\kappa_1 - \rho_x} - A_1 \right)^2 \frac{\varphi_e^2 \bar{\sigma}^2}{1 - \rho_x^2} + A_2^2 \frac{\sigma_w^2}{1 - \nu_1^2} \\
Cov[\Delta c_t^H, r_t^H] &= \left(\frac{1}{\kappa_1 - \rho_x} \right) \left(\frac{1}{\kappa_1 - \rho_x} - A_1 \right) \frac{\varphi_e^2 \bar{\sigma}^2}{1 - \rho_x^2}
\end{aligned}$$

Note that we can verify these by directly computing the variance of wc_t,

$$\begin{aligned}
Var(wc_t) &= Var(A_0 + A_1 x_t + A_2(\sigma_t^2 - \bar{\sigma}^2)) \\
&= A_1^2 \varphi_e^2 \bar{\sigma}^2 + A_2^2 \sigma_w^2 = Var[\Delta c_t^H] - 2Cov[\Delta c_t^H, r_t^H] + Var[r_t^H]
\end{aligned}$$

Numerically, we find

$$\begin{aligned} Var[\Delta c_t^H] &= 0.00586521 \\ Var[r_t^H] &= 0.00261302 \\ Cov[\Delta c_t^H, r_t^H] &= 0.00391013 \\ Var[wc_t] &= 0.000657970 \end{aligned}$$

2. (a) This time we make use of the joint log-normality of r^m and m to obtain

$$\begin{aligned} 0 &= E_t\left[m_{t+1} + r_{t+1}^m\right] + \frac{1}{2}V_t\left[m_{t+1} + r_{t+1}^m\right] \\ &= \theta \ln \beta - \rho\theta(\mu + x_t) \\ &\quad + (\theta - 1)[\kappa_0 + \mu + x_t + A_0 + A_1\rho_x x_t + A_2\nu_1(\sigma_t^2 - \bar{\sigma}^2) - \kappa_1 A_0 \\ &\quad - \kappa_1 A_1 x_t - \kappa_1 A_2(\sigma_t^2 - \bar{\sigma}^2)] \\ &\quad + \mu_d + \phi x_t A_0^m + A_1^m \rho_x x_t + A_2^m \nu_1(\sigma_t^2 - \bar{\sigma}^2) + \kappa_0^m - \kappa_1^m A_0^m \\ &\quad - \kappa_1^m A_1^m x_t - \kappa_1^m A_2^m(\sigma_t^2 - \bar{\sigma}^2) \\ &\quad + \frac{1}{2}[\rho_2\theta^2\sigma_t^2 + (\theta - 1)^2[\sigma_t^2 + A_1^2\varphi_e^2\sigma_t^2 + A_2^2\sigma_w^2] - 2\rho\theta(\theta - 1)\sigma_t^2] \\ &\quad + A_1 A_1^m(\theta - 1)\varphi_e^2\sigma_t^2 + A_2 A_2^m(\theta - 1)\sigma_w^2 \\ &\quad + \frac{1}{2}[\varphi_d^2\sigma_t^2 + (A_1^m\varphi_e\sigma_t)^2 + (A_2^m\sigma_w)^2 \end{aligned}$$

Collecting terms in x_t and $(\sigma_t^2 - \bar{\sigma}^2)$, we again obtain three equations, which we solve to obtain

$$\begin{aligned} A_0^m &= \frac{1}{\kappa_1^m - 1}\Bigg[\theta \ln \beta - \rho\theta\mu + (\theta - 1)(\kappa_0 + \mu + A_0 - \kappa_1 A_0) + \mu_d + \kappa_0^m \\ &\quad + \frac{1}{2}\bar{\sigma}^2\Big[\theta(1 - \rho) - 1]^2 + [(\theta - 1)A_1 + A)1^m]^2\varphi_e^2 + \varphi_d^2\Big] \\ &\quad + \frac{1}{2}[(\theta - 1)A_2 + A_2^m]^2\sigma_w^2]\Bigg] \\ A_1^m &= \frac{1}{\rho_x - \kappa_1^m}[\rho\theta + (1 - \theta)[1 + A_1\rho_x - \kappa_1 A_1] - \phi] \\ A_2^m &= \frac{1}{2(\kappa_1^m - \nu_1)}\Big[[\theta(1 - \rho) - 1]^2 + [(\theta - 1)A_1 + A_1^m]^2\varphi_e^2 + \varphi_d^2 \\ &\quad + 2(\theta - 1)(\nu_1 - \kappa_1)A_2\Big] \end{aligned}$$

(b)

$$\begin{aligned}
E_t\left[r_{t+1}^m\right] - r_t^m &= -\frac{1}{2}V_t\left[r_{t+1}^m\right] - Cov_t[m_{t+1}r_{t+1}^m] \\
&= \sigma_t^2\left[(1-\theta)A_1A_1^m\varphi_e^2 - \frac{1}{2}\varphi_d^2 - \frac{1}{2}(A_1^m)^2\varphi_e^2\right] \\
&\sigma_w^2\left[(1-\theta)A_2A_2^m - \frac{1}{2}(A_2^m)^2\right]
\end{aligned}$$

(c) The inductive argument is the same as in the C-S decomposition. For large H, we obtain

$$\begin{aligned}
pd_t &\approx \frac{\kappa_0^m}{\kappa_1^m - 1} + \sum_{j=1}^{H}(\kappa_1^m)^{-j}E_t\left[\Delta d_{t+j}\right] - \sum_{j=1}^{H}(\kappa_1^m)^{-j}E_t\left[r_{t+j}^m\right] \\
&= \frac{\kappa_0^m}{\kappa_1^m - 1} + \Delta d_t^H - r_t^{mH}
\end{aligned}$$

Approximating the geometric series for H large gives

$$\begin{aligned}
\Delta d_t^H &\approx \frac{\mu d}{\kappa_1^m - 1} + \frac{\phi\rho_x}{\kappa_1^m - \rho_x}x_t \\
r_t^{mH} &\approx \frac{\kappa_0^m + \mu_d}{\kappa_1^m - 1} - A_0^m + \left(\frac{\phi\rho_x}{\kappa_1^m - \rho_x} - A_1^m\right)x_t - A_2^m(\sigma_t^2 - \bar{\sigma}^2)
\end{aligned}$$