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Empirical Asset Pricing - Business 239C - HW#2

1. We begin by evaluating a whole bunch of moments, which we'll make use of later.

$$\begin{aligned} E_t\left[\Delta c_{t+1}\right] &= \mu + x_t \\ E_t\left[x_{t+1}\right] &= \rho_x x_t \\ E_t\left[(\sigma_{t+1}^2 - \bar{\sigma}^2)\right] &= \nu_1(\sigma_t^2 - \bar{\sigma}^2) \\ E_t\left[wc_{t+1}\right] &= A_0 + A_1\rho_x x_t + A_2\nu_1(\sigma_t^2 - \bar{\sigma}^2) \\ E_t\left[r_{t+1}\right] &= \kappa_0 + \mu + (1-\kappa_1)A_0 \\ &\quad + x_t(1 + A_1\rho_x - A_1\kappa_1) \\ &\quad + (\sigma_t^2 - \bar{\sigma}^2)(A_2\nu_1 - \kappa_1 A_1) \\ E_t\left[m_{t+1}\right] &= \theta\ln\beta - \rho\theta(\mu + x_t) + (\theta - 1)E_t\left[r_{t+1}\right] \\ E_t\left[m_{t+1} + r_{t+1}\right] &= \theta\ln\beta - \rho\theta(\mu + x_t) + \theta E_t\left[r_{t+1}\right] \\ &= \theta[\ln\beta + (1-\rho\theta)\mu + \kappa_0 + (1-\kappa_1)A_0 \\ &\quad + x_t(-\rho\theta + 1 + A_1(\rho_x - \kappa_1)) \\ &\quad + (\sigma_t^2 - \bar{\sigma}^2)(A_2\nu_1 - \kappa_1 A_1)] \end{aligned}$$

We'll use the notation σ_{cz} to indicate $Cov(\Delta c_{t+1}, wc_{t+1})$, etc.

$$\begin{aligned} \sigma_{cd} &= 0 \\ \sigma_z^2 &= A_1^2\varphi_e^2\sigma_t^2 + A_2^2\sigma_w^2 \\ \sigma_c^2 &= \sigma_t^2 \\ \sigma_{rc} &= \sigma_c^2 \\ \sigma_r^2 &= (1 + A_1^2\varphi_e^2)\sigma_t^2 + A_2^2\sigma_w^2 \\ \sigma_{rm} &= (\theta-1)\sigma_r^2 - \rho\theta\sigma_c^2 \\ \sigma_m^2 &= \sigma_c^2\rho\theta(\rho\theta - 2\theta + 2) + (\theta-1)^2\sigma_r^2 \\ \sigma_{r+m}^2 &= \sigma_r^2 + 2\sigma_{rm} + \sigma_m^2 \\ &= \sigma_r^2 + 2(\theta-1)\sigma_r^2 - 2\rho\theta\sigma_c^2 + \sigma_c^2\rho\theta(\rho\theta - 2\theta + 2) + (\theta-1)^2\sigma_r^2 \\ &= \rho\theta^2(\rho-2)\sigma_c^2 + \theta^2\sigma_r^2 \\ &= \theta^2((\rho-1)^2 + \varphi_e^2A_1^2)\sigma_t^2 + \theta^2A_2^2\sigma_w^2 \end{aligned}$$

(a) Returns and SDF are log-normal, so $E_t\left[m_{t+1}+r_{t+1}\right]+.5\sigma^2_{r+m}=0$, which gives us an expression of the form

$$A+Bx_t+C(\sigma_t^2-\bar{\sigma}^2)=0$$

For our conjecture about the form of wc_t to hold, this expression must be true for all values of x_t and $(\sigma_t^2-\bar{\sigma}^2)$. It follows that $A=B=B=0$. This yields three equations in A_0, A_1, A_2,

$$\begin{aligned}
0 &= \ln\beta+\kappa_0+(1-\rho)\mu+\frac{1}{2}\theta(1-\rho)^2\bar{\sigma}^2 \\
&\quad +(1-\kappa_1)A_0+\frac{1}{2}\theta A_1^2\varphi_e^2\bar{\sigma}^2+\frac{1}{2}\theta\sigma_w^2A_2^2 \\
0 &= (1-\rho)+A(\rho-\kappa_1) \\
0 &= \frac{1}{2}\theta(1-\rho)^2+\frac{1}{2}\theta A_1^2\varphi_e^2+A_2(\nu-\kappa_1)
\end{aligned}$$

Solving these gives

$$\begin{aligned}
A_0 &= \frac{1}{\kappa_1-1}\left[\ln\beta+\kappa_0+(1-\rho)\mu+\frac{\theta}{2}(1-\rho)^2\bar{\sigma}^2+\frac{\theta}{2}\varphi_e^2\bar{\sigma}^2A_1^2+\frac{\theta}{2}A_2^2\sigma_w^2\right] \\
A_1 &= \frac{\rho-1}{\rho_x-\kappa_1} \\
A_2 &= \frac{\theta(1-\rho)^2}{2(\kappa_1-\nu)}\left(1+\frac{\varphi_e^2}{(\rho_x-\kappa_1)^2}\right)
\end{aligned}$$

(b) We used a numerical solver from Matlab's optimization toolkit to solve the above equations for the unspecified parameters. We obtained the following:

$$\begin{aligned}
\kappa_0 &= 0.0198 \\
\kappa_1 &= 1.0029 \\
A_0 &= 5.8515 \\
A_1 &= 5.1624 \\
A_2 &= -175.1020
\end{aligned}$$

(c) Using the Euler Equation and the log-normality of m and r,

$$R_{f,t}=\frac{1}{E_t\left[M_{t+1}\right]}=e^{-E_t[m_{t+1}]-\frac{1}{2}V_t[m_{t+1}]}$$

Taking logs gives the risk free rate,

$$r_{f,t} = -E_t\left[m_{t+1}\right] - \frac{1}{2}V_t\left[m_{t+1}\right]$$

and subtracting this from the Euler Equation gives excess returns,

$$E_t\left[r_{t+1} - r_{f,t}\right] = -\frac{1}{2}V_t\left[r_{t+1}\right] - Cov_t[r_{t+1}, m_{t+1}]$$

(d) We prove by induction that

$$wc_t = \sum_{j=1}^{H} \kappa_1^{-j}(\Delta c_{t+j} - r_{t+j} + \kappa_0) + \kappa_1^{-H} wc_{t+H}$$

The claim holds for $H = 1$ because

$$\begin{aligned}\kappa_1 wc_t &= \Delta c_{t+1} + wc_{t+1} + \kappa_0 - r_{t+1} \\ \Rightarrow wc_t &= \kappa^{-1}(\Delta c_{t+1} + wc_{t+1} + \kappa_0) + \kappa_1^{-1} wc_{t+H}\end{aligned}$$

If the claim holds for $N = n$, then we simply expand the final term to yield the claim for $N = n + 1$,

$$\begin{aligned}wc_t &= \sum_{j=1}^{n} \kappa_1^{-j}(\Delta c_{t+j} - r_{t+j} + \kappa_0) + \kappa_1^{-n} wc_{t+n} \\ &= \sum_{j=1}^{n} \kappa_1^{-j}(\Delta c_{t+j} - r_{t+j} + \kappa_0) \\ &\quad + \kappa_1^{-(n+1)}(\Delta c_{t+n+1} + \kappa_0 - r_{t+n+1} + wc_{t+n+1}) \\ &= \sum_{j=1}^{n+1} \kappa_1^{-j}(\Delta c_{t+j} - r_{t+j} + \kappa_0) + \kappa_1^{-(n+1)} wc_{t+n+1})\end{aligned}$$

Now we take the limit of our claim as $H \to \infty$. By the transversality condition, $E_t\left[wc_{t+H}\right] \to E[wc_t]$, which is finite, so for large H, the last term vanishes, and the κ_0 term approaches an infinite geometric series. That is,

$$\begin{aligned}wc_t &\approx E_t\left[\sum_{j=1}^{H} \kappa_1^{-j}(\Delta c_{t+j} + r_{t+j})\right] + \frac{\kappa_0}{\kappa_1}\frac{1}{1 - \frac{1}{\kappa_1}} \\ &= \frac{\kappa_0}{\kappa_1 - 1} + \Delta c_t^H + r_t^H\end{aligned}$$

(e) To evaluate the variance decomposition, we must first solve for Δc_t^H and r_t^H.

$$\begin{aligned}\Delta c_t^H &= \sum_{j=1}^{H} \kappa_1^{-j} E_t\left[\Delta c_{t+j}\right] \\ &= \sum_{j=1}^{H} \kappa_1^{-j}(\mu + E x_{t+j-1}) \\ &= \sum_{j=1}^{H} \kappa_1^{-j}(\mu + \rho_x^{j-1} x_t)\end{aligned}$$

For large H and $\kappa_1 > \rho_x$, this power series converges to

$$\Delta c_t^H \approx \frac{1}{\kappa_1 - 1}\mu + \frac{1}{\kappa_1 - \rho_x} x_t$$

A similar power series summation yields

$$r_t^H \approx \frac{\kappa_0 - \mu}{\kappa_1 - 1} - A_0 + x_t \left(\frac{1}{\kappa_1 - \rho_x} - A_1 \right) + (\sigma_t^2 - \bar{\sigma}^2) A_2$$

Now we can compute the unconditional variances,

$$\begin{aligned}Var[\Delta c_t^H] &= \left(\frac{1}{\kappa_1 - \rho_x} \right)^2 \sigma_x^2 = \left(\frac{1}{\kappa_1 - \rho_x} \right)^2 \varphi_e^2 \bar{\sigma}^2 \\ Var[r_t^H] &= \left(\frac{1}{\kappa_1 - \rho_x} - A_1 \right)^2 \varphi_e^2 \bar{\sigma}^2 + A_2^2 \sigma_w^2 \\ Cov[\Delta c_t^H, r_t^H] &= \left(\frac{1}{\kappa_1 - \rho_x} \right) \left(\frac{1}{\kappa_1 - \rho_x} - A_1 \right) \varphi_e^2 \bar{\sigma}^2\end{aligned}$$

We can verify our expressions by directly computing the variance of wc_t,

$$\begin{aligned}Var(wc_t) &= Var(A_0 + A_1 x_t + A_2(\sigma_t^2 - \bar{\sigma}^2)) \\ &= A_1^2 \varphi_e^2 \bar{\sigma}^2 + A_2^2 \sigma_w^2 = Var[\Delta c_t^H] - 2Cov[\Delta c_t^H, r_t^H] + Var[r_t^H]\end{aligned}$$