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General Equilibrium - HW #1

1. **Gainers and Losers in Equilibrium**

Note that preferences are identical and homothetic, so we may use a representative agent to find PTE's.

(a) In each region, the PTE allocations are symmetric:

$$\omega_{A_1} = \omega_{A_2} = (2,1)$$
$$\omega_{B_1} = \omega_{B_2} = (1,2)$$

To see this in region A, substitute $y = 2z$. This yields symmetric preferences and endowments, $u = xz/2$ and $\omega = (4,4)$, so the RA sets equal prices: $(p_x, p_z) = (1,1)$. Equivalently,

$$\mathbf{p} = (p_x, p_y) = (1,2)$$

With these prices, A_1 and A_2 have equal wealth, so each consumes the half the aggregate endowment, obtaining $u = 2$. The analysis is identical for region B, but with $\mathbf{p} = (2,1)$.

(b) Aggregate endowment is symmetric: $\omega = (6,6)$, so $\mathbf{p} = (1,1)$. Type 1 agents have twice as much wealth as Type 2 agents, so they consume twice as much of the aggregate endowment:

$$\text{Type 1: } c = (2,2); u = 4$$
$$\text{Type 1: } c = (1,1); u = 1$$

(c-d) Any allocation that gives 1/4 of the total wealth to each agent results in a PTE with each agent consuming (1.5,1.5) and utility $u = 2.25$.

(e) The allocation in (b) is Pareto Efficient, as is the allocation in (d). Free trade does not guarantee a "fair" equilibrium, nor one that maximizes aggregate utility.

2. **Replica Invariance**

(a) Any utility $u_1 + u_2 \leq 100$ is feasible. From the given endowment, there are no mutually advantageous trades.

(b) In $\mathcal{E}^2$ a more efficient allocation is possible:

$$\text{Type 1: } c = (50, 50)$$
$$\text{Type 2: } c_1 = (100, 0),\ c_2 = (0, 100)$$

The Type 2 agents now have increased utility, $u = 100$, and the Type 1 still have $u = 50$ as in the original endowment.

(c) In a PTE, each agent maximizes utility subject to his wealth constraint. If there is a PTE in $\mathcal{E}$, then the same prices must yield a PTE in $\mathcal{E}^2$.

As we'll show in (e), the allocation in (b) is a PTE with $\mathbf{p} = 1$. Any PTE in $\mathcal{E}$ must therefore have the same prices. We also know that a PTE in $\mathcal{E}$ must allocation (50,50) for each agent, as this endowment is already Pareto Optimal.

This allocation, however, is not an equilibrium with $\mathbf{p} = 1$, because the Type 2 agents can increase utility by trading. For example, $c = (49, 51)$ yields $u = 51$. It follows that there is no PTE in $\mathcal{E}$.

(d) Preferences for the Type 1 agent do not exhibit local non-satiation. For example, at both (50,50) and (50,51), his utility is 50. Preferences for the Type 2 agent are not convex. For example, both (100,0) and (0,100) are strictly preferred to (50,50).

(e) For $\mathcal{F}^k$, give (50,50) to each Type 1 agent, and (100,0) to half the Type 2's, (0,100) to the other half. No agent can be made better off without increasing the gross total $x + y$ of his consumption. Thus, we cannot improve upon this allocation without increasing the gross aggregate endowment, $\omega_x + \omega_y$. The allocation is therefore efficient for all k.

(f) With $\mathbf{p} = 1$, each agent's bugdet constraint is $x + y = 100$. The Type 1 agents maximize utility at (50,50), while the Type 2 agents maximize at the boundaries (0,100) or (100,0). In the allocation in (e), each agent consumes one of these utility maximizing bundles. The allocation is therefore a PTE.

3. **Demand Theory with Quasi-linear Utility**

(a) Denote indirect utility as $V(p,w)$. Let any w, w' be given and denote the corresponding optimal non-money consumption as x, x' respectively. Since x is optimal for w, we have

$$\begin{aligned} U(x, w-px) &\geq U(x', w-px') \\ &= v(x') + w - px' \\ &= U(x', w'-px') + (w-w') \end{aligned}$$

Comparing the first and last lines, we have

$$V(p,w) - w \geq V(p,w') - w',$$

but our choice of w and w' was arbitrary, so the reverse inequality also holds. Thus, for all w, w', we have

$$V(p,w) - w = V(p,w') - w',$$

So $\alpha(p) = V(p,w) - w$ is independent of w, as desired.

(b) Denote the expenditure function as $C(p,U)$. From duality theory, we know

$$U = V\big(p, C(p,U)\big)$$

Applying the form from (a), this becomes

$$U = \alpha(p) + C(p,U),$$

so expenditure is simply

$$C(p,U) = -\alpha(p) - U$$

(c) Given any w, w', we show that we achieve the same utility with either x or x':

$$\begin{aligned} U(x', w-px') &= v(x') + w - px' \\ &= v(x') + w' - px' + (w-w') \\ &= U(x', w') + w - w' \\ &= V(p,w') - w' + w \\ &= V(p,w) - w + w \qquad \text{(from (a))} \\ &= V(p,w) \\ &= U(x, w-px) \end{aligned}$$

Thus, x' is optimal for wealth w, so optimal non-money consumption is independent of wealth, as desired.

(d) In equilibrium, we have

$$\partial v / \partial x_i = p_i$$

By the envelope theorem, differentiating with respect to p_i gives

$$\frac{\partial^2 v}{\partial x_i^2} \frac{\partial x_i}{\partial p_i} = 1$$

Assuming $v(x)$ is concave, it's second differential is negative, so $\partial x_i / \partial p_i$ is also negative, as is own price elasticity.

4. **Demand/Supply, Inverse Demand/Suppy and Indirect Utility/Profit**

(a) $(1 \Rightarrow 2)$: Suppose $(z, m) \in D(p)$. Then $pz + m = 0$ and

$$v(z) + m \geq v(z') + m'$$

for all (m', z') with $pz' + m' = 0$. So we have

$$v(z) - pz \geq v(z') - pz'$$

for all $z' \in Z$. By definition, $p \in \partial v(z)$.

$(2 \Rightarrow 3)$: If $p \in \partial v(z)$, then for all $z' \in Z$, we have

$$v(z) - pz \geq v(z') - pz'.$$

That is,

$$\begin{aligned} v(z) - pz &= \max\{v(z') - pz'\} \\ &= \max\{v(z') + m' \mid pz' + m' = 0\} \end{aligned}$$

That is, $v(z) - pz$ is the maximum utility feasible with zero total wealth. This is the definition of the indirect utility function $v^*(p)$.

$(3 \Rightarrow 1)$: By definition, $v^*(p)$ is the maximum utilty acheivable with zero net wealth. That is,

$$v^*(p) = max\{v(z') + m' \mid pz' + m = 0\}.$$

Since $v(z) - pz = v^*(p)$, this implies $(z, -pz) \in D(p)$.

(b) The revenue from selling z is $r = -pz$, which we previously denoted m. The firm's profit is therefore

$$\pi(p, z) = -c(z) - pz = v(z) + r$$

In this notation, the conditions above become

(1) $(z, r) \in \text{argmax}_{z,r}\{v(z) + r \mid pz + r = 0\}$
(2) $-p \in \partial c(z)$
(3) $\pi^*(p) = -c(z) - pz$